Sistem Komunikasi 1

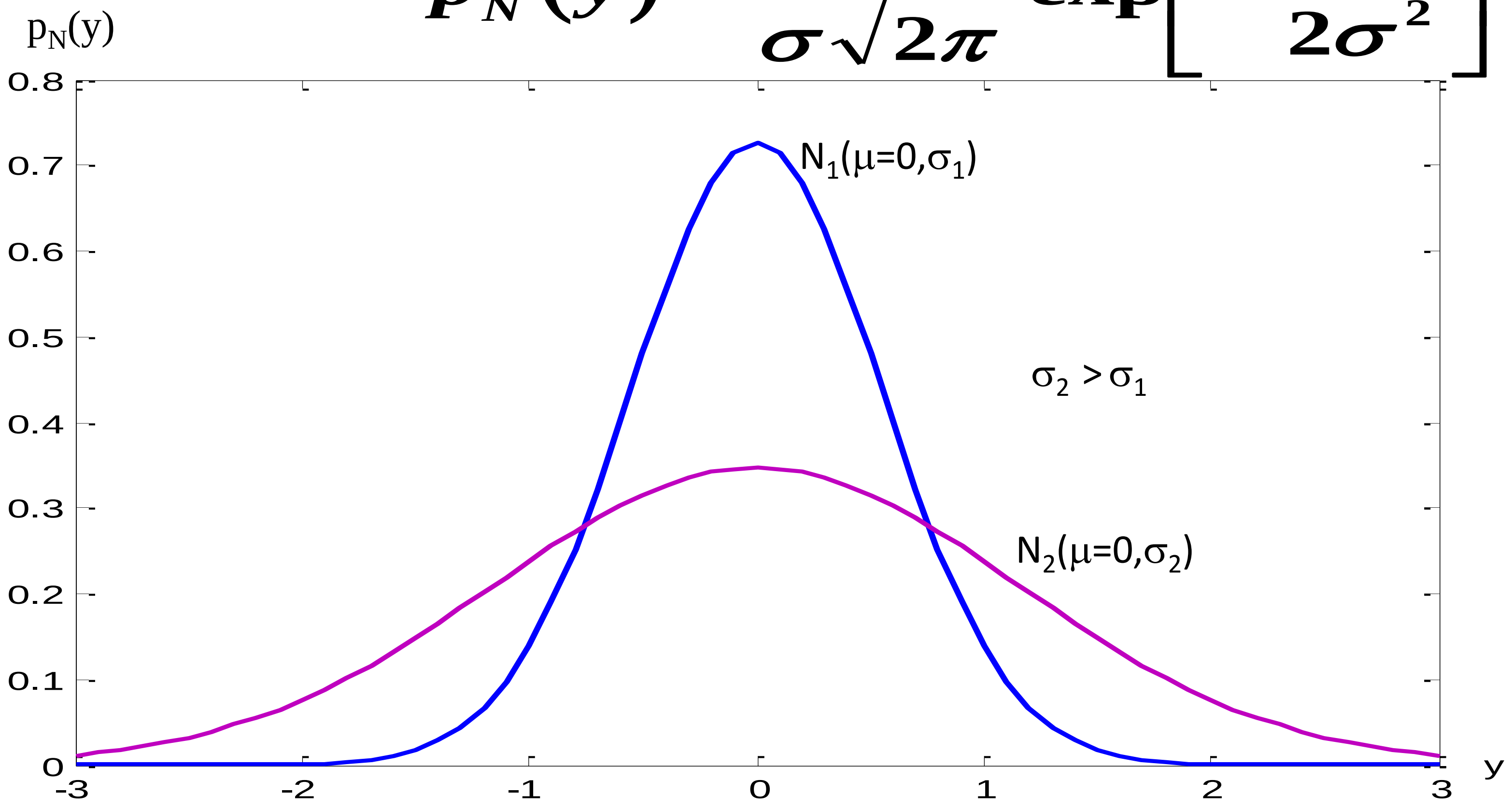
Bab 10
Probabilitas Error

Review Probabilitas dan Statistik

Gaussian/Normal

- **<u>Normal Distribution:</u>** Completely characterized by mean (μ) and variance (σ^2)

$$p_N(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{y^2}{2\sigma^2}\right]$$



Gaussian: Rapidly Dropping Tail Probability!

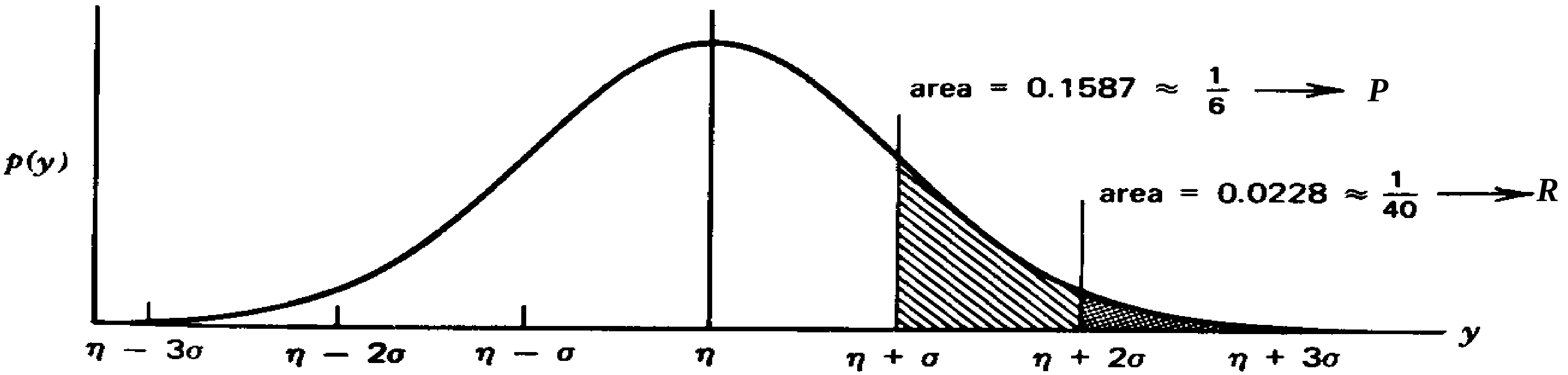


FIGURE 2.12. Tail areas of the normal distribution.

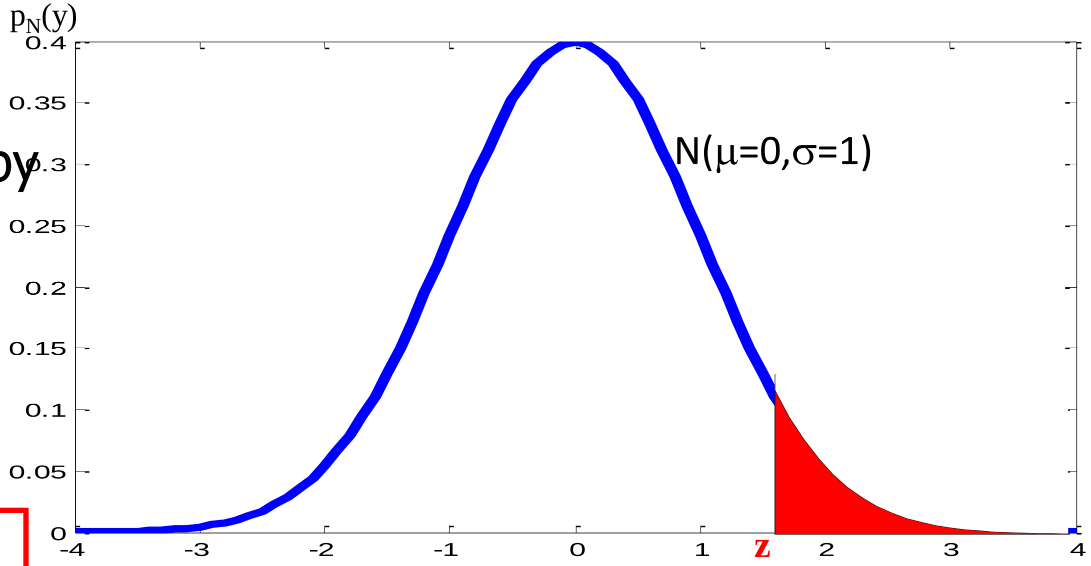
$$P = \frac{1}{\sigma\sqrt{2\pi}} \int_{\eta+\sigma}^{\infty} \exp\left[-\frac{(y-\eta)^2}{2\sigma^2}\right] dy$$

$$R = \frac{1}{\sigma\sqrt{2\pi}} \int_{\eta+2\sigma}^{\infty} \exp\left[-\frac{(y-\eta)^2}{2\sigma^2}\right] dy$$

Review Probabilitas dan Statistik

Gaussian/Normal

- **Normal Distribution:** Completely characterized by mean (μ=0) and variance (σ^2=1)

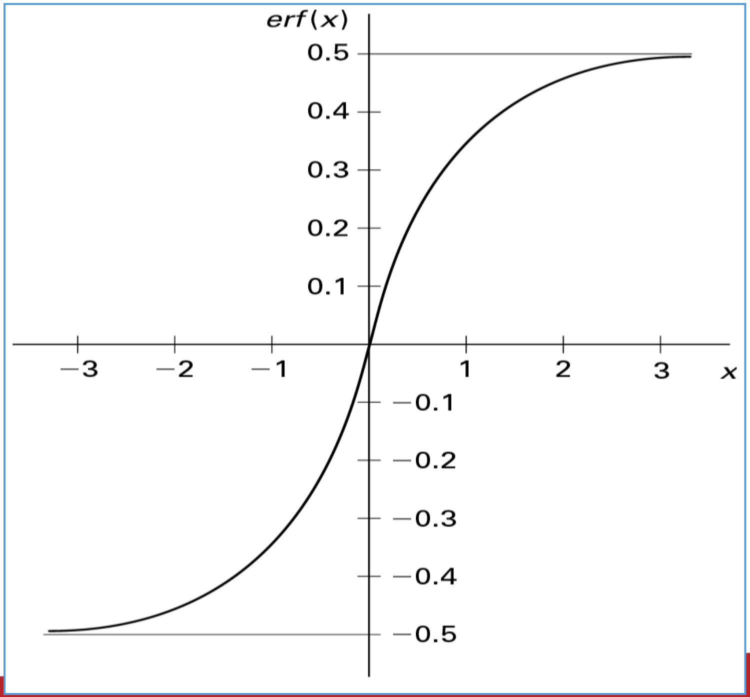


- **Q-function:** one-sided tail of normal pdf

$$Q(z) \triangleq p(y>z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

- **erfc():** two-sided tail.
- So:

$$Q(z) = \frac{1}{2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right).$$

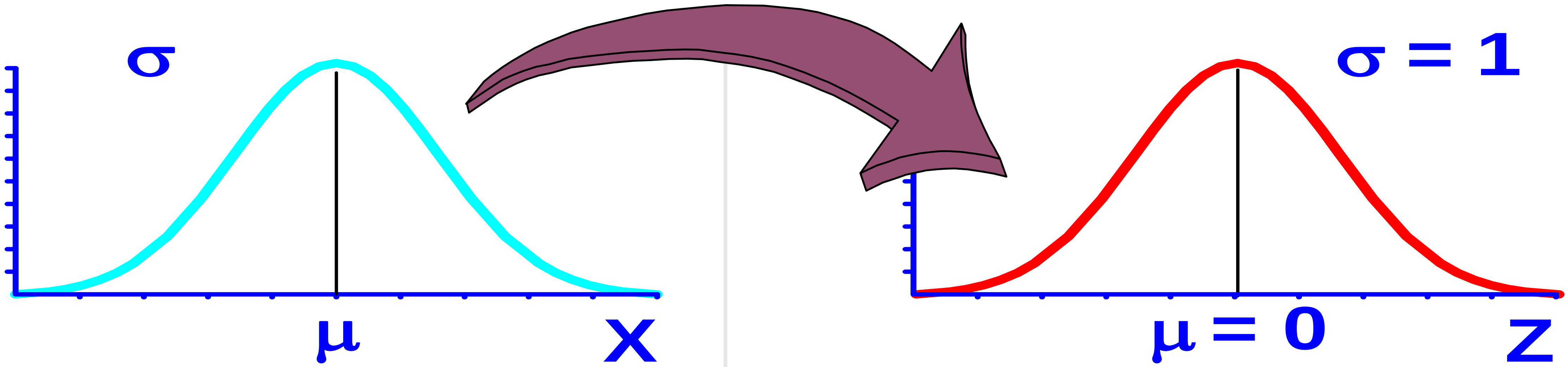


Standardize the Normal Distribution

$$z = \frac{x - \mu}{\sigma}$$

Normal Distribution

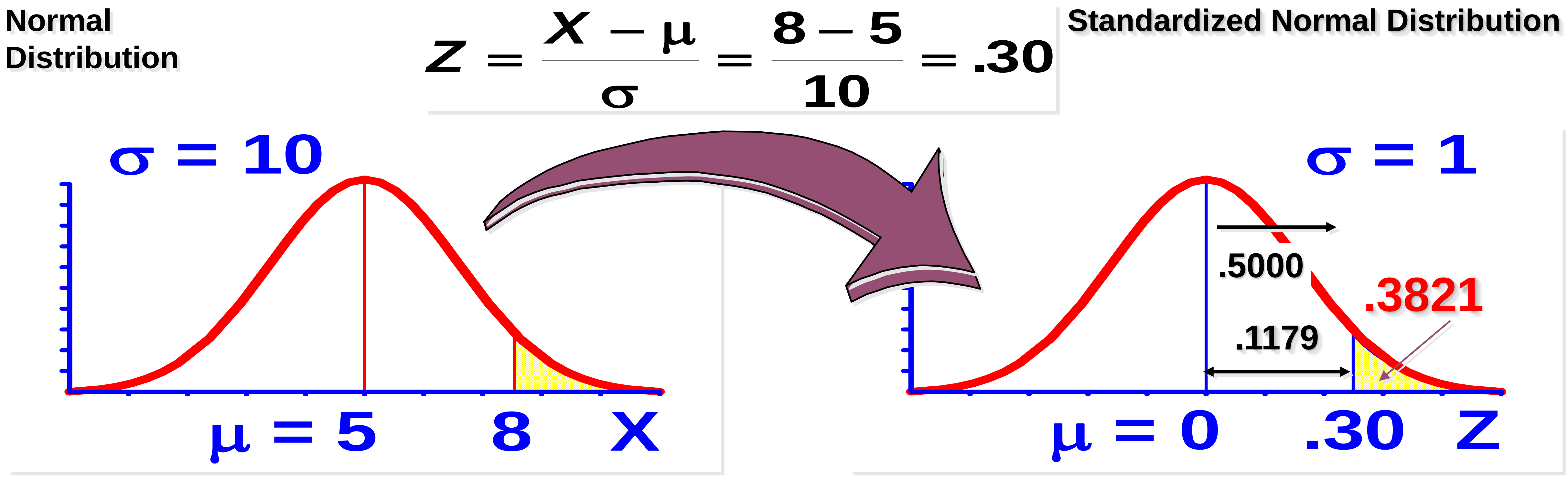
Standardized Normal Distribution



One table!

Example P($X \geq 8$)

$$Z = \frac{X - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$



Shaded area exaggerated

PERTANYAAN ! Luas daerah yang diarsir = 0.3821 = Q(??); ?? = **0.3**

Q-function: Tail of Normal Distribution

$Q(z) = P(Z > z) = 1 - P[Z < z]$

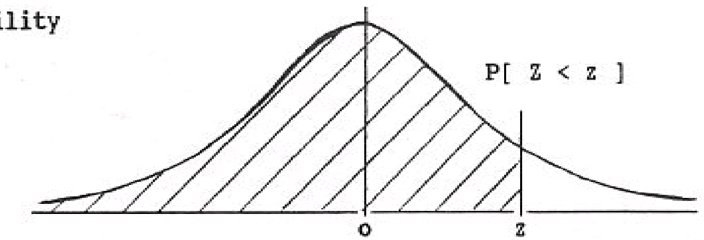
STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z

i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

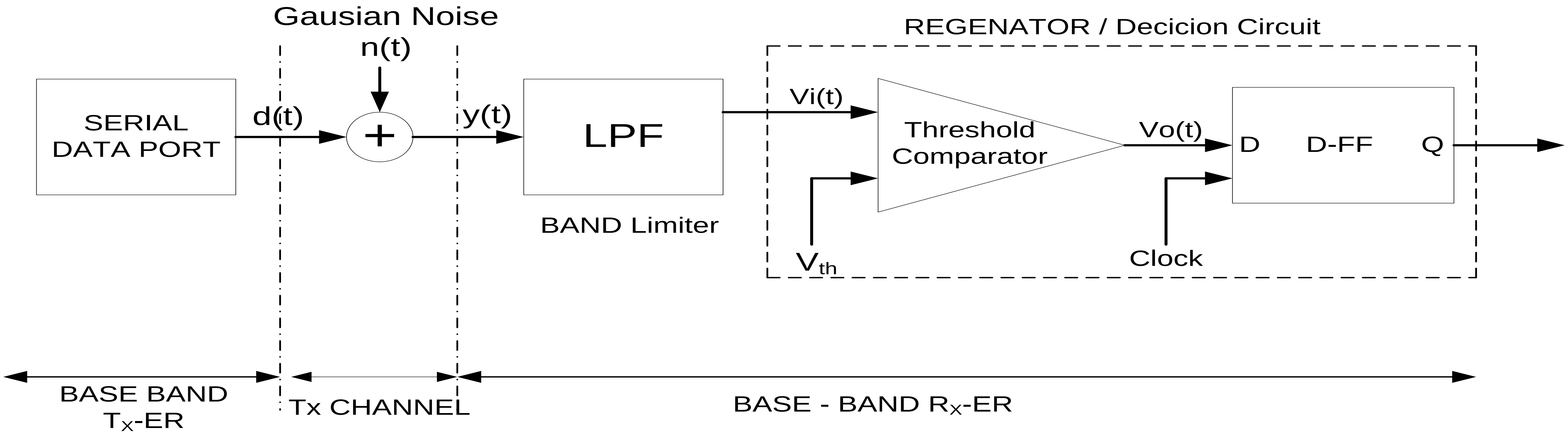
z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

Tabel Q-function

TABLE B.1 Complementary Error Function $Q(x) = \int_x^{\infty} (1/\sqrt{2\pi}) \exp(-u^2/2)\, du$

	Q(x)									
x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2168	0.2148
0.8	0.2169	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

Baseband Digital Transmission Link



Sinyal Terima + AWGN

original message
d(t)

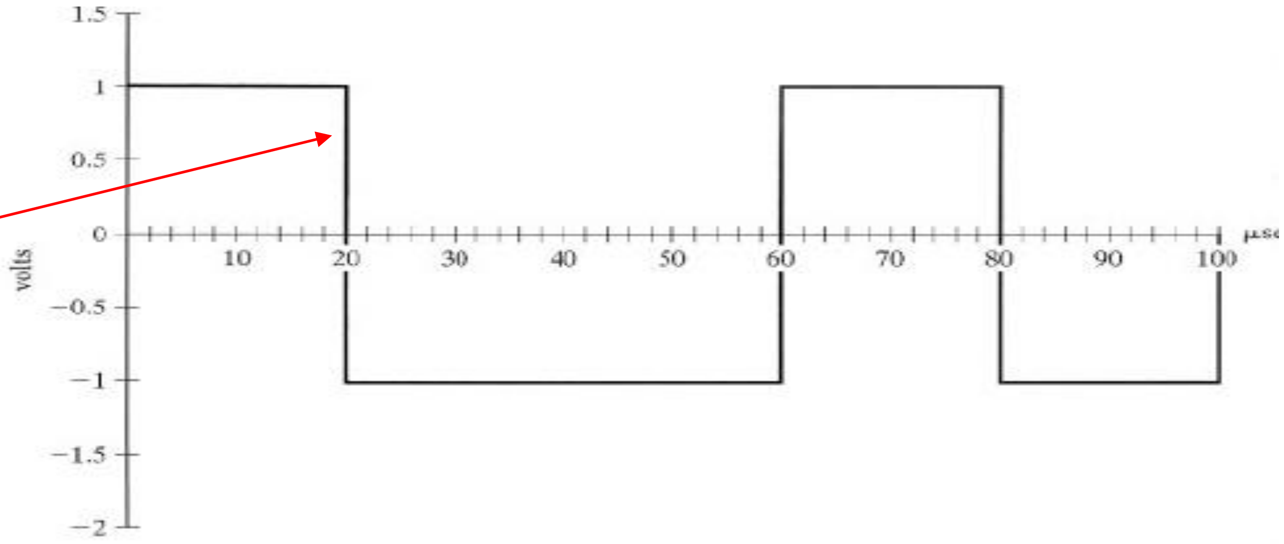
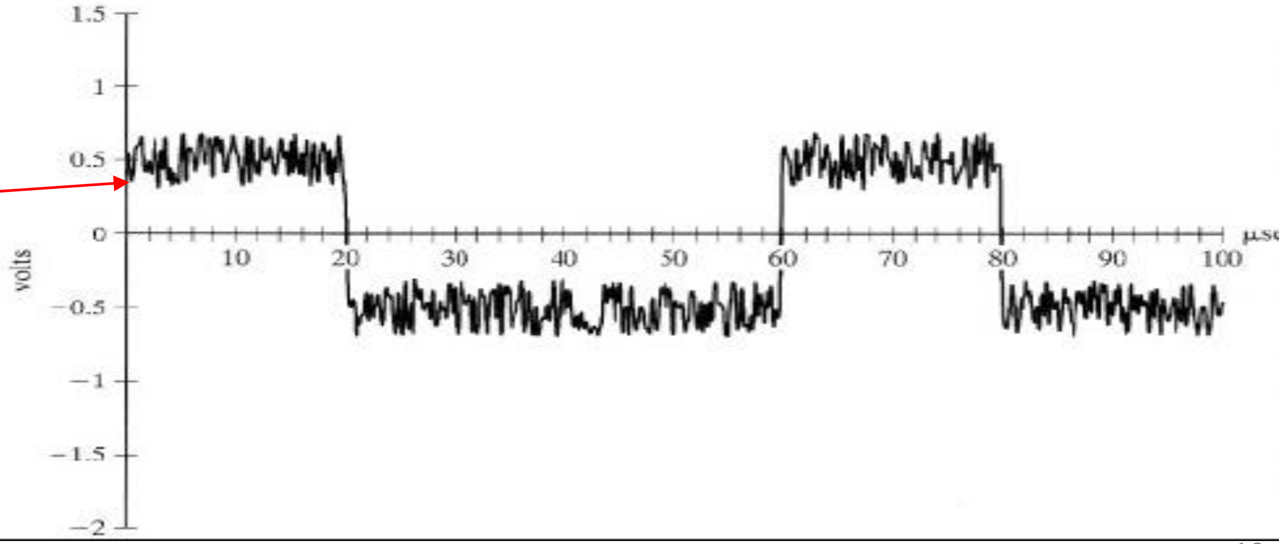


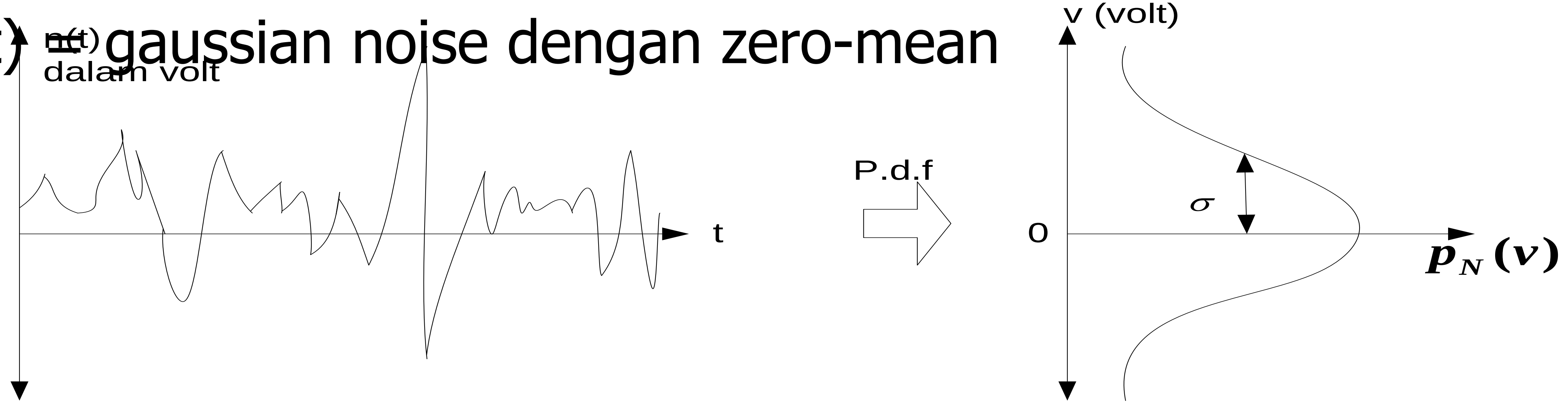
Figure 3-23a Transmitted signal for "10010" using rectangular pulses.

received wave
y(t)=d(t)+n(t)



AWGN

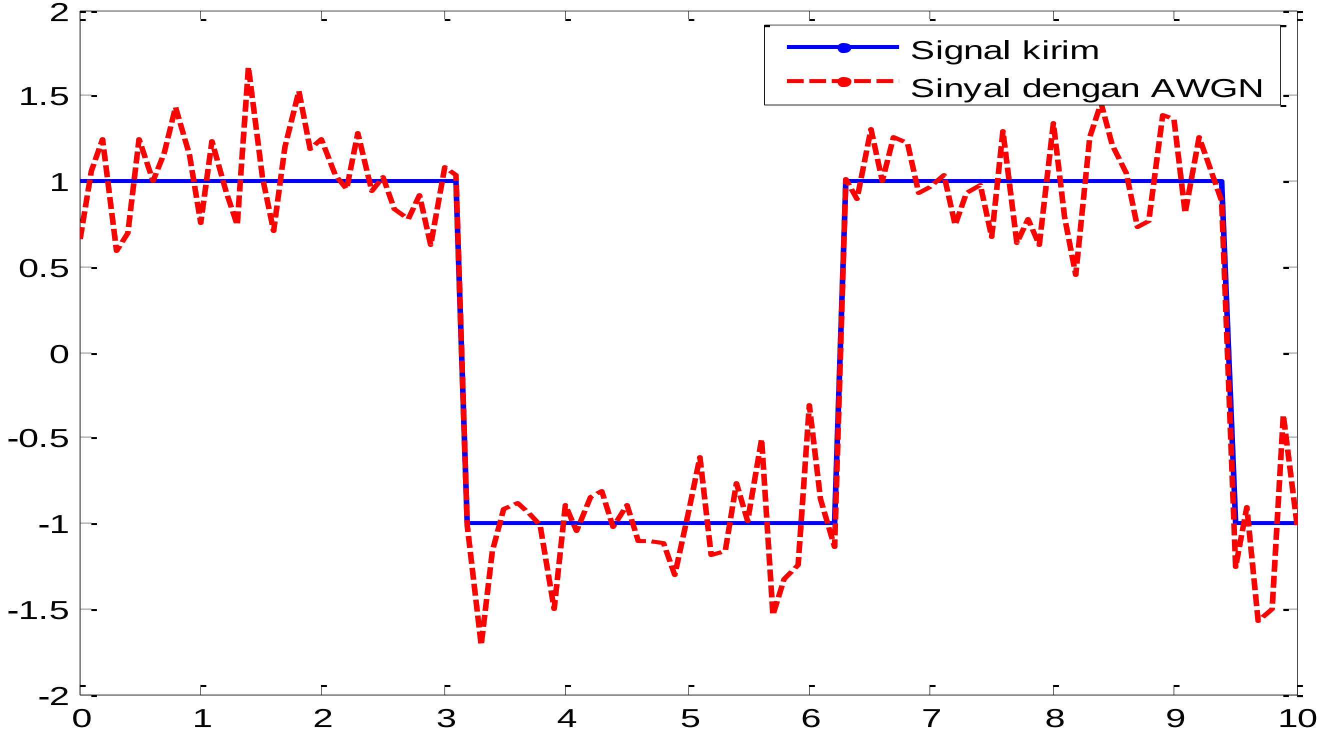
- n(t) = gaussian noise dengan zero-mean

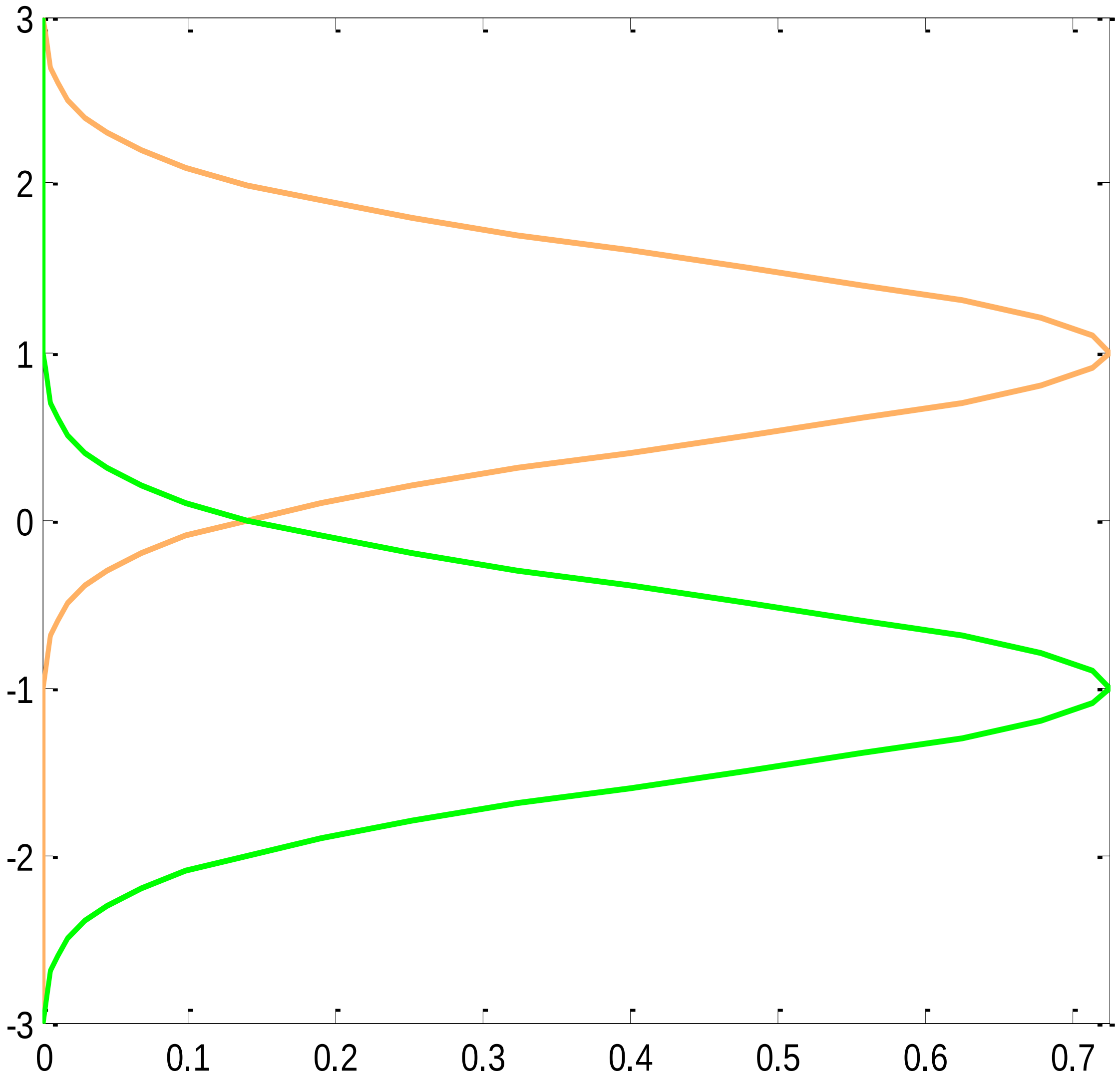


$$p_N(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{v^2}{2\sigma^2}\right]$$

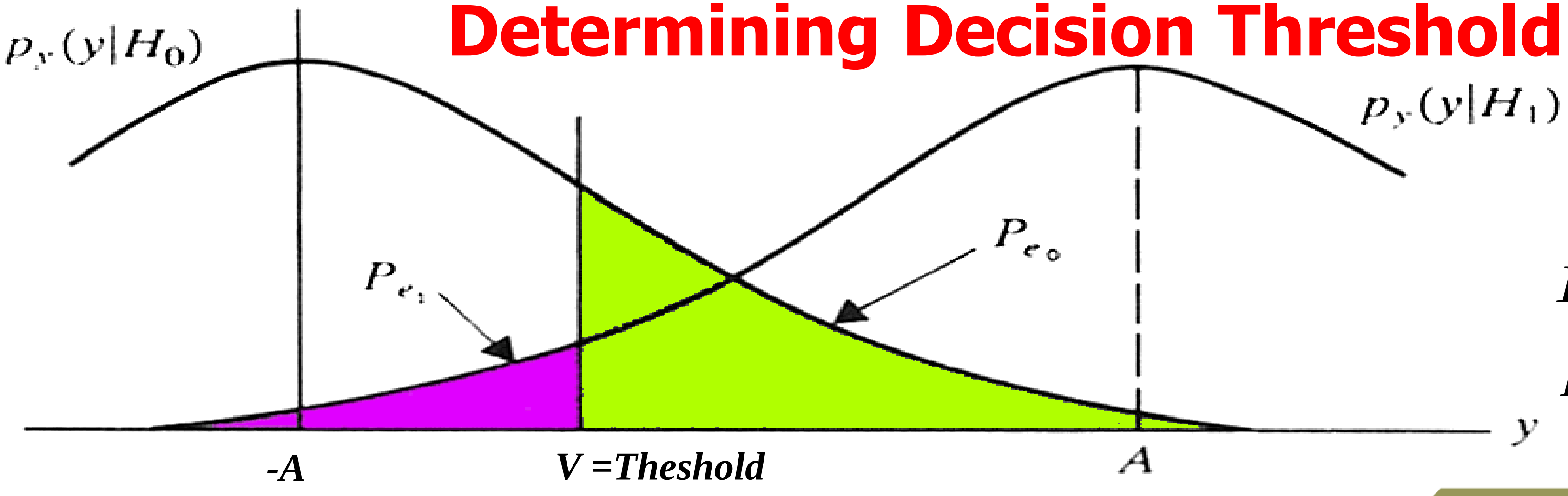
- σ = standar deviasi = tegangan effektive noise

Gangguan Noise Terhadap Sinyal Digital





Determining Decision Threshold



$$H_0 : a_k = 0, Y = -A + n$$
$$p_Y(y/H_0) = p_N(y + A)$$
$$H_1 : a_k = 1, Y = A + n$$
$$p_Y(y|H_1) = p_N(y - A)$$

The comparator implements decision rule:

Choose Ho (a_k=0) if $Y<V$
Choose H1 (a_k=1) if $Y>V$

$$p_{e1} \equiv P(Y < V | H_1) = \int_{-\infty}^{V} p_Y(y|H_1)dy$$

$$p_{eo} \equiv P(Y > V | H_0) = \int_{V}^{\infty} p_Y(y|H_0)dy$$

Transmitted '0' but detected as '1'

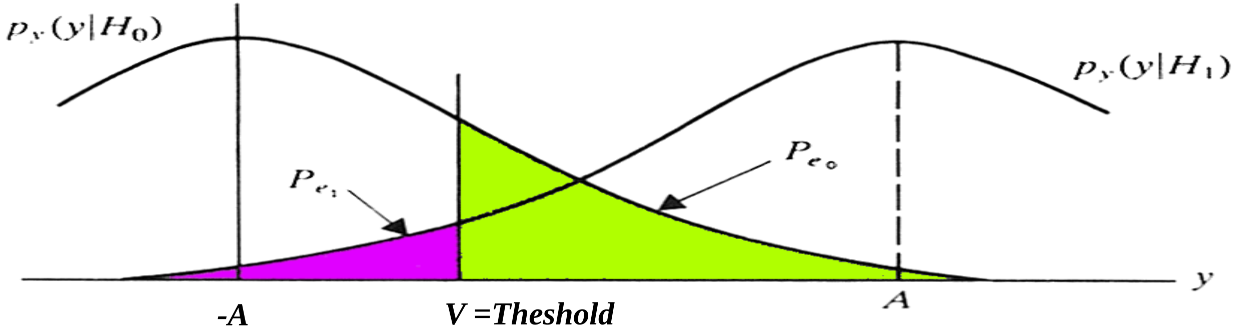
Average error error probability: $P_e = P_0 P_{e0} + P_1 P_{e1}$

$$P_0 = P_1 = 1/2 \Rightarrow P_e = ½(P_{e0} + P_{e1})$$

Channel noise is Gaussian with the pdf:

$$p_N(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{y^2}{2\sigma^2}\right]$$

Error rate and Q-function



$$P_0 = P_1 = 1/2 \Rightarrow P_e = \tfrac{1}{2}(P_{e0} + P_{e1})$$

V threshold = 0

$$P_{e0} = \int_V^{\infty} p_N(y)dy$$

$$p_e = p_{e0} = \frac{1}{\sigma\sqrt{2\pi}} \int_{V=0}^{\infty} \exp\left[-\frac{(y+A)^2}{2\sigma^2}\right]dy$$

This can be expressed by using the Q-function

$$Q(z) \triangleq p(x > z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

by

$$p_{e0} = \int_V^{\infty} p_N(y)dy = p_e = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right)$$

Baseband Binary Error Rate in Terms of Pulse Shape and S/N

setting *V=0* yields then

$$p_e = ½(p_{e0} + p_{e1}) = p_{e0} = p_{e1} \Rightarrow p_e = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{\frac{S}{N}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

for polar, **rectangular** NRZ [-A,A] bits

Probability of occurrence

Signal power:

$$S = \frac{1}{2}A^2 + \frac{1}{2}(-A)^2 = A^2$$

Noise power:

$$N = \sigma^2 = \eta.BW_N = N_0.\frac{R_b}{2} = N_0.\frac{1}{2T_b}$$

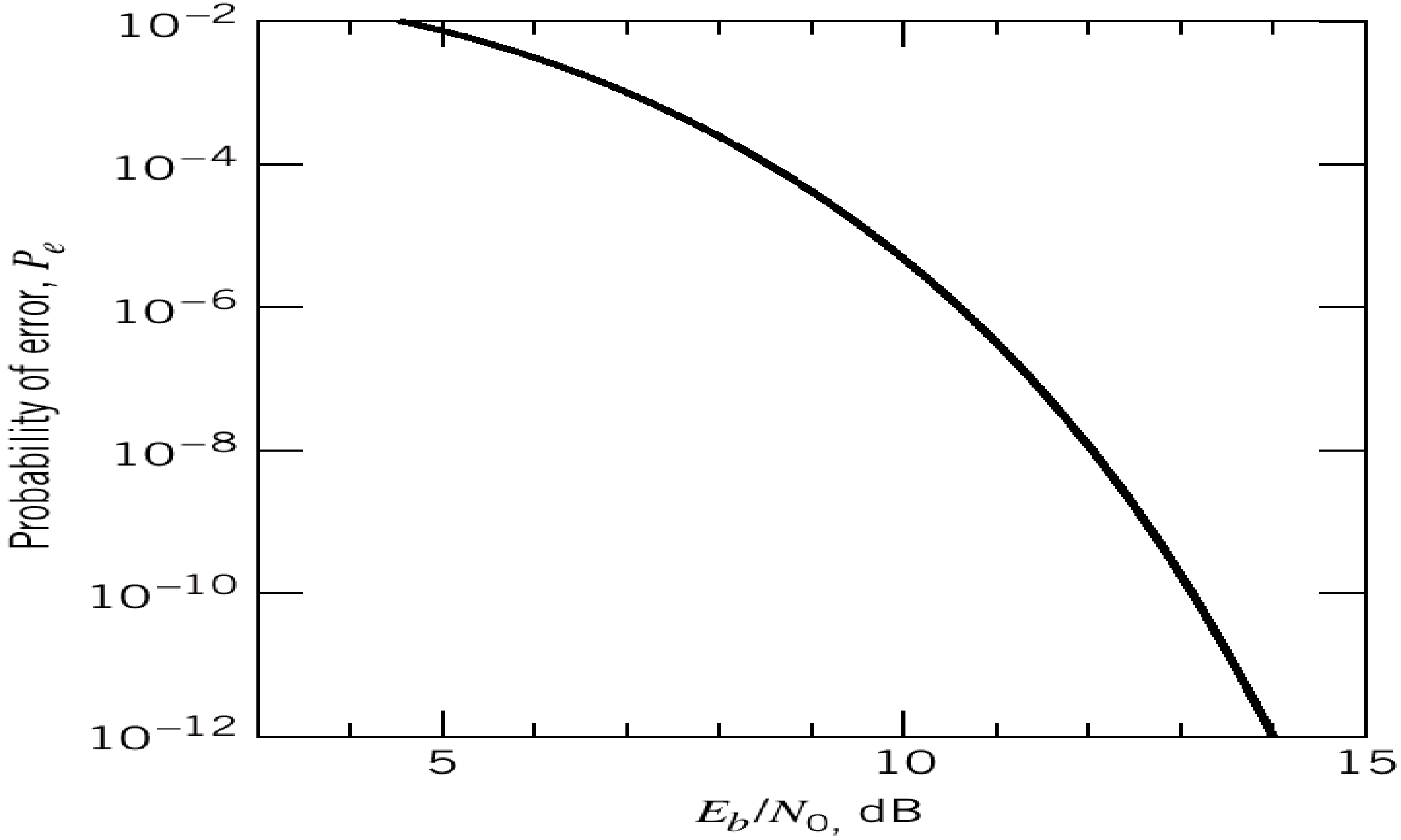
Energy Bit to Noise Spectral Density Ratio

$$\frac{E_b}{N_0} = \frac{S.T_b}{N/BW_N} = \frac{S.T_b}{N/(Rb/2)} = \frac{S.T_b}{N}.\frac{R_b}{2} = \frac{S.T_b}{N}.\frac{1}{2.T_b} = \frac{1}{2}.\frac{S}{N}$$

Note that $BW_N = \frac{R_b}{2}$ (BW pulse shapping filter)

When $p_0 = p_1 = 1/2$, the value of V that minimizes the probability of error is V = 0.

$$p_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{S}{N}}\right)$$

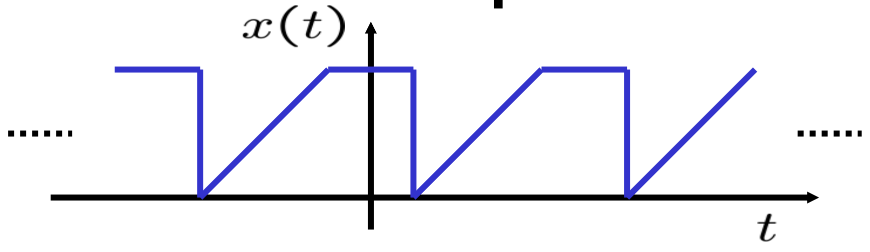


Classification of signals

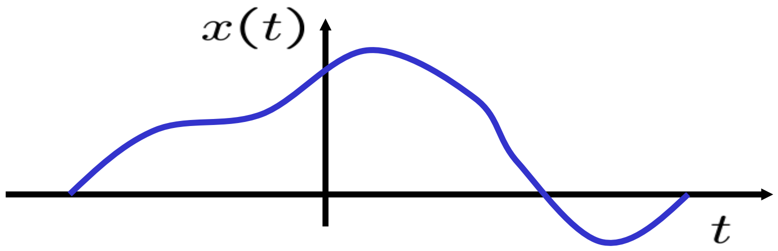
- Deterministic and random signals
  - Deterministic signal: No uncertainty with respect to the signal value at any time.
  - Random signal: Some degree of uncertainty in signal values before it actually occurs.
    - Thermal noise in electronic circuits due to the random movement of electrons
    - Reflection of radio waves from different layers of ionosphere

Classification of signals …

- Periodic and non-periodic signals

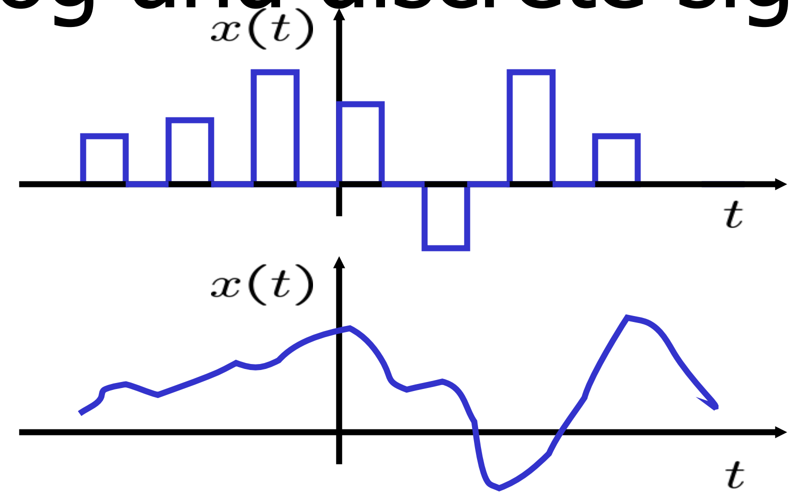


A periodic signal

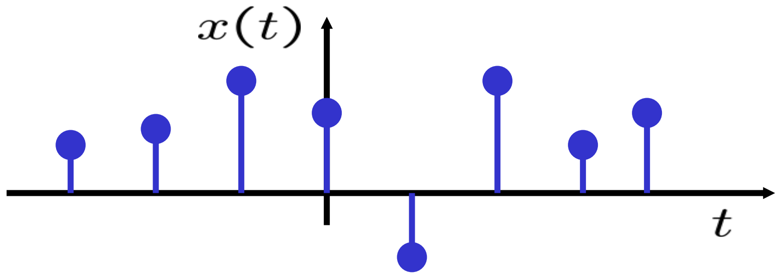


A non-periodic signal

- Analog and discrete signals



Analog signals



A discrete signal

Classification of signals ..

- Energy and power signals
  - A signal is an energy signal if, and only if, it has nonzero but finite energy for all time:

$$(0 < E_x < \infty)$$

$$E_x = \lim_{T\to\infty} \int_{T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

  - A signal is a power signal if, and only if, it has finite but nonzero power for all time:

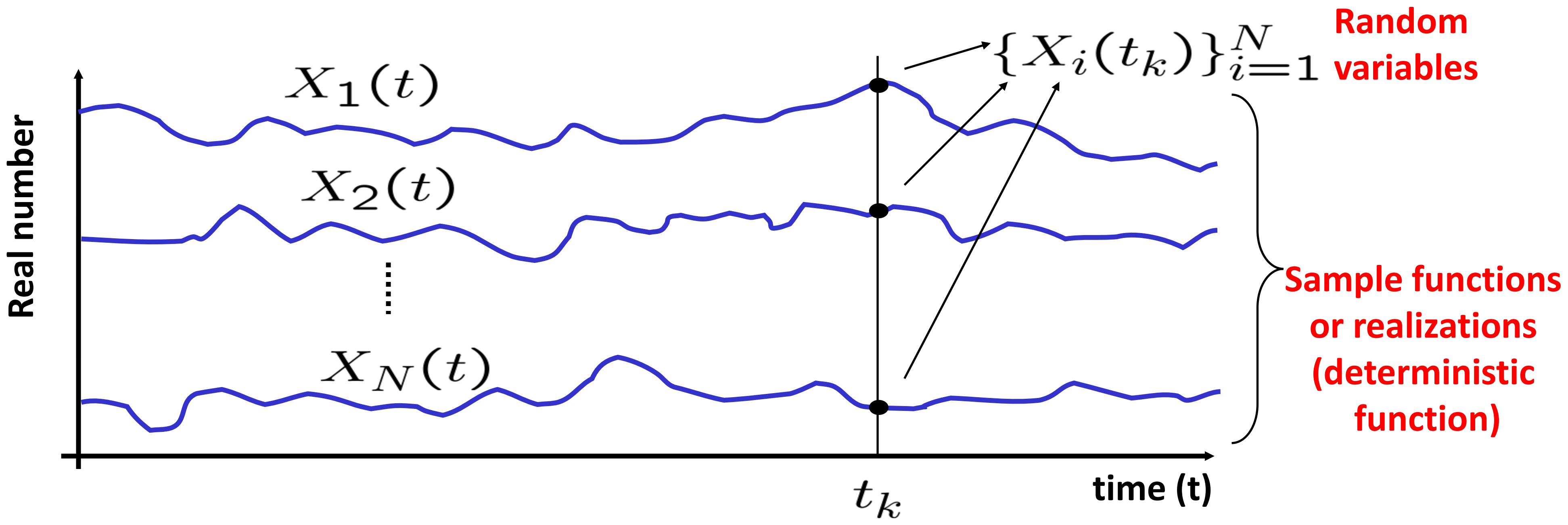
$$(0 < P_x < \infty)$$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} |x(t)|^2 dt$$

  - General rule: Periodic and random signals are power signals. Signals that are both deterministic and non-periodic are energy signals.

Random process

- A random process is a collection of time functions, or signals, corresponding to various outcomes of a random experiment. For each outcome, there exists a deterministic function, which is called a sample function or a realization.



Random process …

- Strictly stationary: If none of the statistics of the random process are affected by a shift in the time origin.
- Wide sense stationary (WSS): If the mean and autocorrelation function do not change with a shift in the origin time.
- Cyclostationary: If the mean and autocorrelation function are periodic in time.
- Ergodic process: A random process is ergodic in mean and autocorrelation, if

$$m_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)dt$$

and

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)X^*(t-\tau)dt$$

, respectively.

Autocorrelation

- Autocorrelation of an energy signal

$$R_x(\tau) = x(\tau) \star x^*(-\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a power signal

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x^*(t-\tau)dt$$

  - For a periodic signal:

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt$$

- Autocorrelation of a random signal

$$R_X(t_i, t_j) = \mathrm{E}[X(t_i)X^*(t_j)]$$

  - For a WSS process:

$$R_X(\tau) = \mathrm{E}[X(t)X^*(t-\tau)]$$

Spectral density

- Energy signals:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad X(f) = \mathcal{F}[x(t)]$$

  - Energy spectral density (ESD):

$$\Psi_x(f) = |X(f)|^2$$

- Power signals:

$$P_x = \frac{1}{T_0} \int_{T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \qquad \{c_n\} = \mathcal{F}[x(t)]$$

  - Power spectral density (PSD):

$$G_x(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0) \qquad f_0 = 1/T_0$$

- Random process:
  - Power spectral density (PSD):

$$G_X(f) = \mathcal{F}[R_X(\tau)]$$

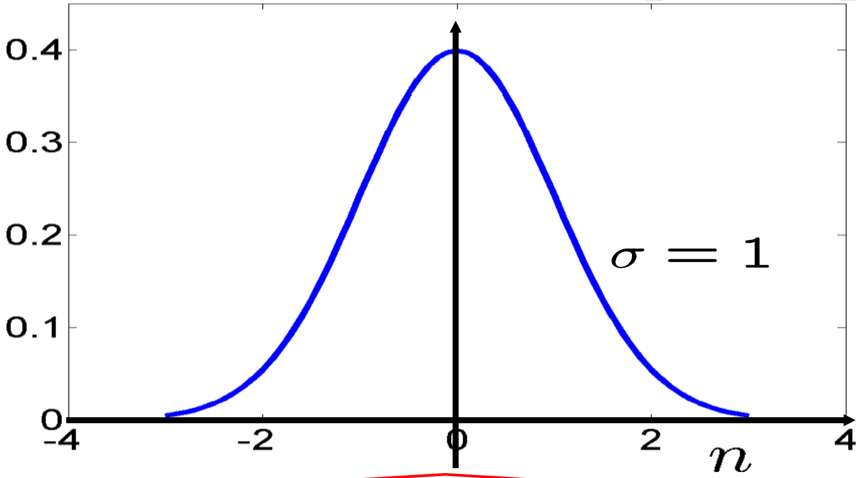
Properties of an autocorrelation function

- For real-valued (and WSS in case of random signals):
  1. Autocorrelation and spectral density form a Fourier transform pair.
  2. Autocorrelation is symmetric around zero.
  3. Its maximum value occurs at the origin.
  4. Its value at the origin is equal to the average power or energy.

Noise in communication systems

- Thermal noise is described by a zero-mean Gaussian random process, n(t).
- Its PSD is flat, hence, it is called white noise.

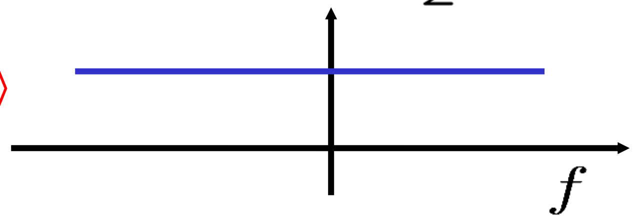
$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{n^2}{2\sigma^2}\right]$$



Probability density function

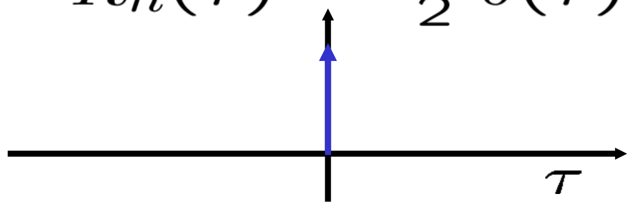
$$G_n(f) = \frac{N_0}{2} \text{ [w/Hz]}$$

Power spectral density

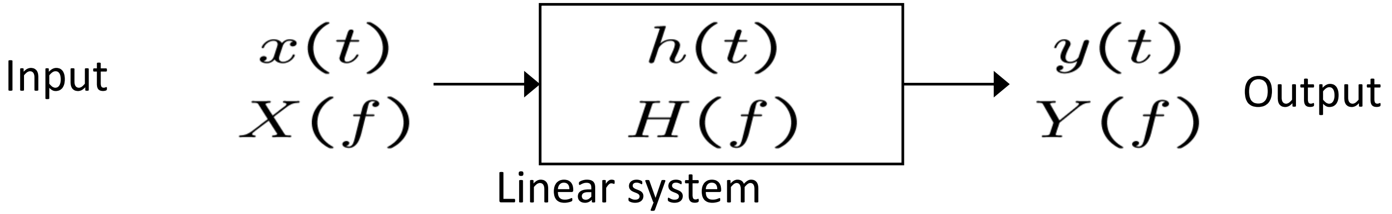


$$R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

Autocorrelation function



Signal transmission through linear systems



- Deterministic signals: $$Y(f) = X(f)H(f)$$
- Random signals: $$G_Y(f) = G_X(f)|H(f)|^2$$

- Ideal distortionless transmission:

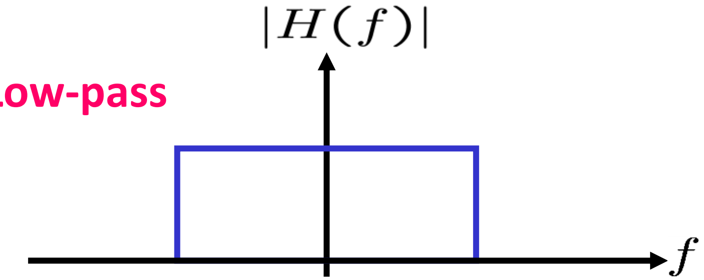
  All the frequency components of the signal not only arrive with an identical time delay, but also are amplified or attenuated equally.

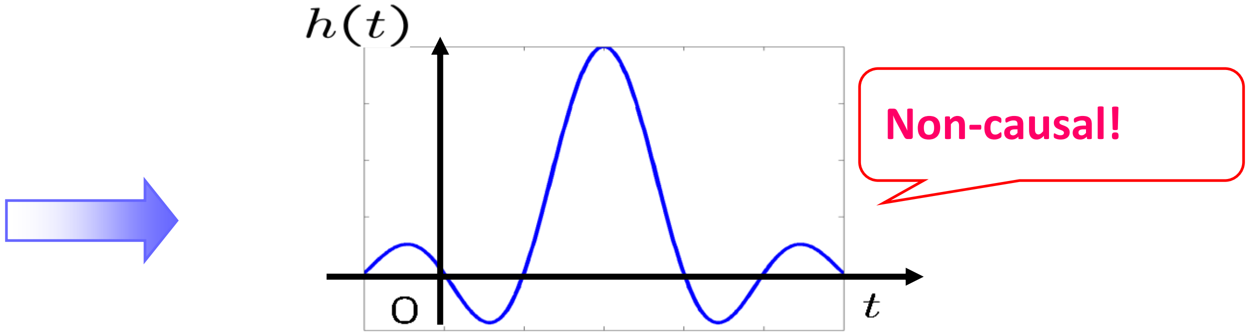
$$y(t) = Kx(t - t_0) \text{ or } H(f) = Ke^{-j2\pi f t_0}$$

Signal transmission … - cont'd

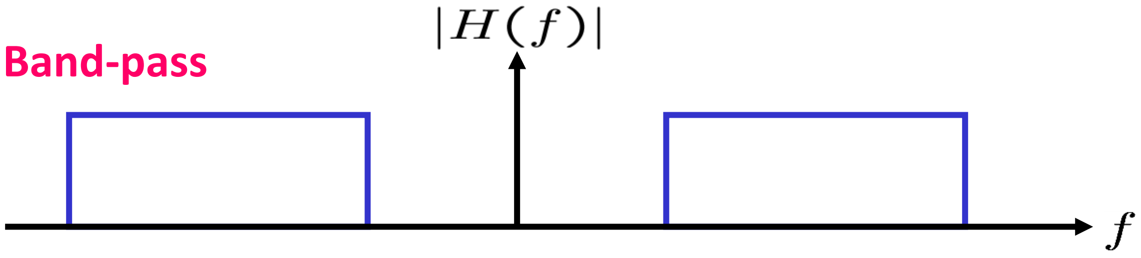
- Ideal filters:

Low-pass

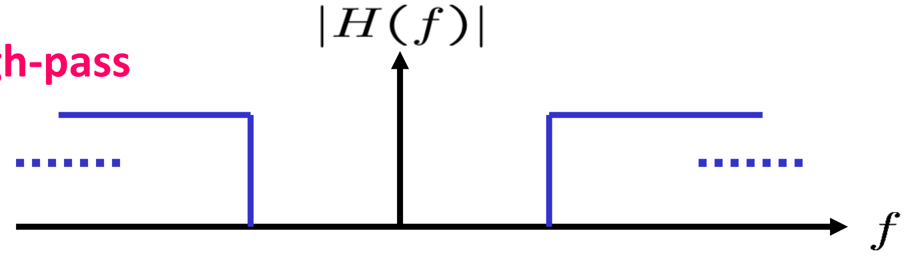




Band-pass



High-pass



- Realizable filters:

RC filters

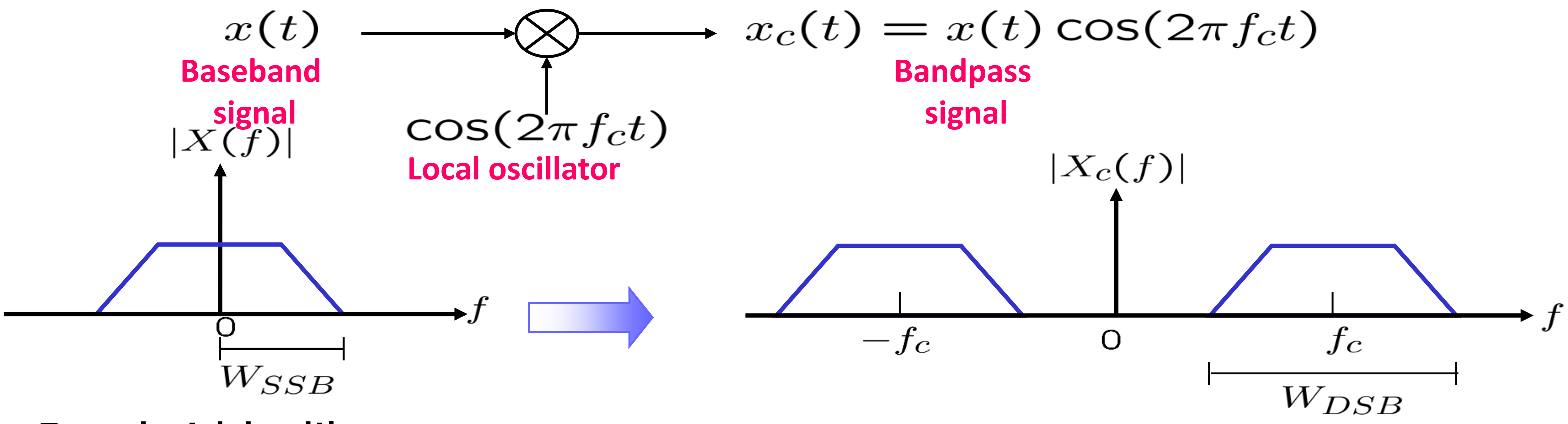
$$H(f) = \frac{1}{1 + j2\pi f RC}$$

Butterworth filter

$$|H_n(f)| = \frac{1}{\sqrt{1 + (f/f_u)^{2n}}}$$

Bandwidth of signal

- Baseband versus bandpass:

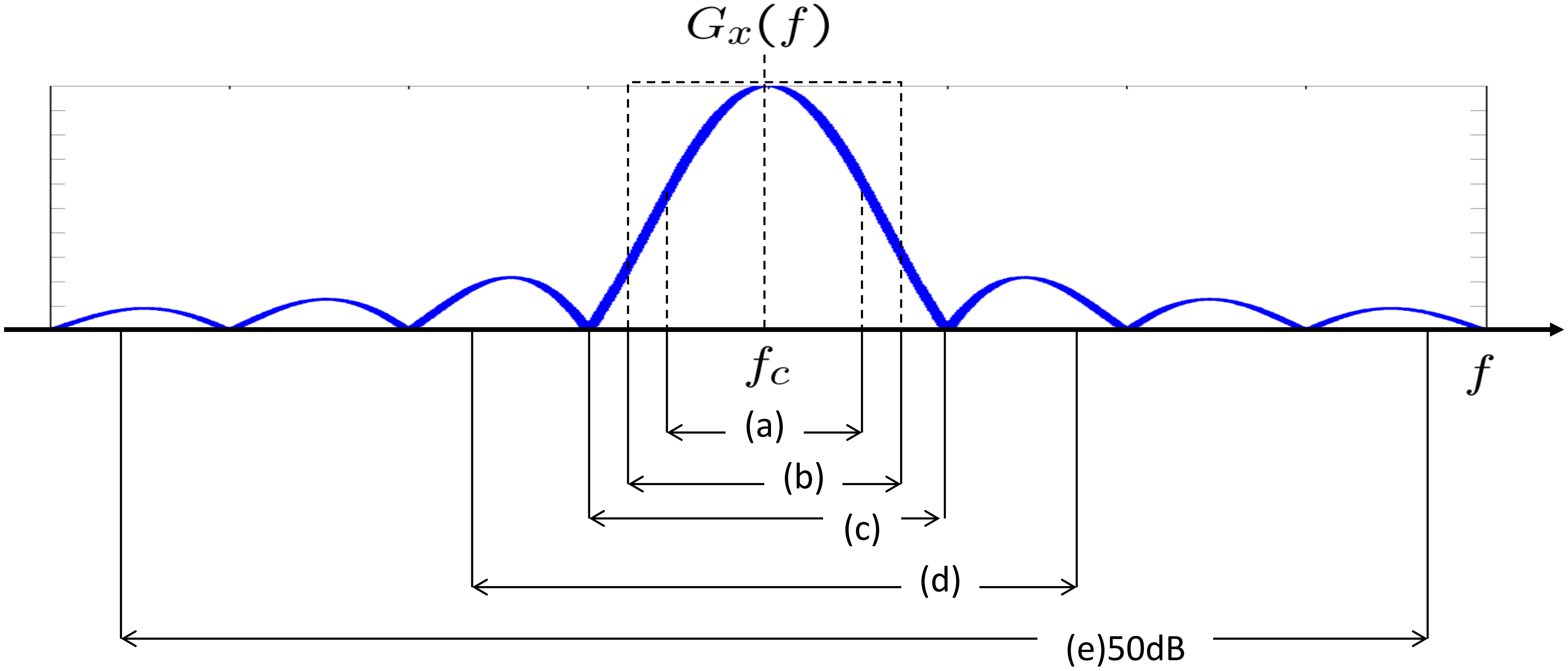


- Bandwidth dilemma:
  - Bandlimited signals are not realizable!
  - Realizable signals have infinite bandwidth!

Bandwidth of signal – cont'd

- Different definition of bandwidth:

a) Half-power bandwidth
b) Noise equivalent bandwidth
c) Null-to-null bandwidth
d) Fractional power containment bandwidth
e) Bounded power spectral density
f) Absolute bandwidth



End of Module 10