Introduction to Proof Methods Part 1: Direct Proof and Indirect Proof Using Contrapositive

Mathematical Logic – First Term 2023-2024

MZI

School of Computing
Telkom University

SoC Tel-U

November-December 2023

Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 1), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 4), 5th Edition, 2018, by S. S. Epp.
3. Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
4. Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

Contents

Terminology: Theorem, Lemma, Proposition, and Proof

Theorem, Lemma, and Proposition

Theorem

A *theorem* usually denotes a (mathematical) statement that can be shown to be true and somewhat important or (very) useful. Theorems can also be referred to as mathematical facts or results.

Lemma

A lemma ("little theorem") usually denotes a less important theorem/ mathematical fact/result that is important in the proof of other results/ more important fact (lemma is rarely stand-alone). Complicated proofs are usually easier to understand when they are proved using series of "lemma".

Proposition

A proposition usually denotes a statement that is less important than theorems. Sometimes propositions is stand-alone. However, propositions are also used in the inference of more complicated proofs.

Mathematical Proofs

Proof: a valid argument that establish the truth of particular theorems, lemmas, or proposition. The statements used in proof can include axioms or postulates (*the statements that are assumed to be true*). A proof is obtained from a valid inference from a collection of premises. We usually mark the end of a proof with following symbol: $\square$, $\blacksquare$, or $\mathbf{Q.E.D}$.

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)	$a^2 - b^2 = ab - b^2$	By subtracting both sides of (2) with b^2.
(4)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)	$a^2 - b^2 = ab - b^2$	By subtracting both sides of (2) with b^2.
(4)	$(a-b)(a+b) = b(a-b)$	Algebraic factorization of each sides of (3).
(5)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)	$a^2 - b^2 = ab - b^2$	By subtracting both sides of (2) with b^2.
(4)	$(a-b)(a+b) = b(a-b)$	Algebraic factorization of each sides of (3).
(5)	$a + b = b$	By dividing each sides of (4) with $a - b$.
(6)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)	$a^2 - b^2 = ab - b^2$	By subtracting both sides of (2) with b^2.
(4)	$(a - b)(a + b) = b(a - b)$	Algebraic factorization of each sides of (3).
(5)	$a + b = b$	By dividing each sides of (4) with $a - b$.
(6)	$2b = b$	By substituting a with b in (5), because we have $a = b$ in (1).
(7)		

Why do we need to learn mathematical proofs?

Reason 1: to avoid deriving incorrect mathematical statements.

"Theorem"

$1 = 2$

"Proof"

Suppose a and b are two equal integers, then

	Step	Reason
(1)	$a = b$	Given assumption.
(2)	$a^2 = ab$	By multiplying both sides of (1) with a.
(3)	$a^2 - b^2 = ab - b^2$	By subtracting both sides of (2) with b^2.
(4)	$(a - b)(a + b) = b(a - b)$	Algebraic factorization of each sides of (3).
(5)	$a + b = b$	By dividing each sides of (4) with $a - b$.
(6)	$2b = b$	By substituting a with b in (5), because we have $a = b$ in (1).
(7)	$2 = 1$	By dividing each sides of (6) with b. $\square$

Reason 2: to guarantee that our arguments apply in general setting.

Theorem

The sum of two odd integers is an even integer.

"Proof"

Reason 2: to guarantee that our arguments apply in general setting.

Theorem

The sum of two odd integers is an even integer.

"Proof"

Take two arbitrary odd integers, suppose these integers are a and b. Assume that $a = 1$ and $b = 3$. It is obvious that a and b are odd integers. We have $a + b = 1 + 3 = 4$, hence $a + b$ is an even integer. Based on this argument, the sum of two odd integers is an even integer. □

Reason 3: to understand the correct and effective elementary proof methods.
Reason 4:

Reason 3: to understand the correct and effective elementary proof methods.
Reason 4: because proof methods are used in more advanced course such as:

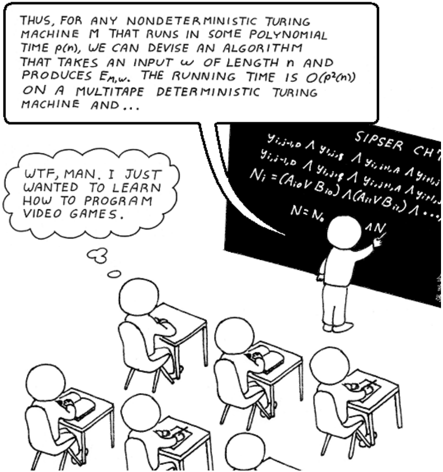
Reason 3: to understand the correct and effective elementary proof methods.
Reason 4: because proof methods are used in more advanced course such as:

- proving the *correctness* of an algorithm – which will be learned in Complexity Analysis of Algorithm class (a compulsory course for undergraduate informatics major)

Reason 3: to understand the correct and effective elementary proof methods.
Reason 4: because proof methods are used in more advanced course such as:

- proving the *correctness* of an algorithm – which will be learned in Complexity Analysis of Algorithm class (a compulsory course for undergraduate informatics major)
- providing undeniable facts in particular systems – which is used extensively in several elective course such as Cryptography and Formal Methods.

Mathematical proving in computer science...



Source: ABSTRUSE GOOSE.

.

Benefits for Computer Science Major

The methods of proof are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include:

1. verifying that computer programs are correct,
2. establishing that operating systems are secure,
3. making inferences in artificial intelligence, and
4. showing that system specifications are consistent.

Contents

Assumption and Prerequisite

Assumption and Prerequisite

- Familiarity with algebraic manipulations taught in high school.
- Properties of equality ($=$), i.e.: (1) $A = A$, (2) if $A = B$ then $B = A$, and (3) if $A = B$ and $B = C$, then $A = C$.
- There is no integer between 0 and 1.
- The set of integers is closed under addition, subtraction, and multiplication. This means that for all integers a and b, the numbers $a + b$, $a - b$, and ab are also integers.

Contents

Sentence and Language in Theorem and Proofs

Theorems and mathematical proofs can be expressed formally using predicate logic (as we learned in the inference system for predicate logic). However, formal proofs of useful theorems can be extremely long and hard to follow. In practice, proofs of theorems designed for human consumption are almost always informal proofs, which usually described in natural language sentences (e.g.: English, Bahasa Indonesia, etc.).

Many theorems assert that particular property holds for all elements in a domain (such as integers or real numbers). In these theorem, a universal quantification is usually not explicitly mentioned (although sometimes we need to use universal quantification for precisely explaining a statement). In the proofs of such theorems, universal instantiation (taking an arbitrary element c of a particular domain) is used implicitly.

For example, the following theorem:

Theorem

Many theorems assert that particular property holds for all elements in a domain (such as integers or real numbers). In these theorem, a universal quantification is usually not explicitly mentioned (although sometimes we need to use universal quantification for precisely explaining a statement). In the proofs of such theorems, universal instantiation (taking an arbitrary element c of a particular domain) is used implicitly.

For example, the following theorem:

Theorem

If $x > y$, where x and y are positive real numbers greater than 1, then $x^2 > y^2$.

actually means:

Theorem

Many theorems assert that particular property holds for all elements in a domain (such as integers or real numbers). In these theorem, a universal quantification is usually not explicitly mentioned (although sometimes we need to use universal quantification for precisely explaining a statement). In the proofs of such theorems, universal instantiation (taking an arbitrary element c of a particular domain) is used implicitly.

For example, the following theorem:

Theorem

If $x > y$, where x and y are positive real numbers greater than 1, then $x^2 > y^2$.

actually means:

Theorem

For all positive real numbers x and y that are greater than 1, if $x > y$, then $x^2 > y^2$.

Contents

Direct Proofs

Simple Direct Proof

Suppose we have a theorem (or lemma/ proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.
A direct proof of this theorem is constructed as follows:

Direct Proofs

Simple Direct Proof

Suppose we have a theorem (or lemma/ proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.
A direct proof of this theorem is constructed as follows:

- first, assume that p is **true**;

Direct Proofs

Simple Direct Proof

Suppose we have a theorem (or lemma/ proposition) in a conditional statement: "if p then q", or $p \to q$.
A direct proof of this theorem is constructed as follows:

- first, assume that p is **true**;
- construct subsequent statements using rules of inference until... q is **true**.

Quantified Direct Proof

Suppose we have a theorem (or lemma/ proposition) of the form: "for all $x \in D$, if $P(x)$ then $Q(x)$", or $\forall x \, (P(x) \to Q(x))$.
A direct proof of this theorem is constructed as follows:

Direct Proofs

Simple Direct Proof

Suppose we have a theorem (or lemma/ proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.
A direct proof of this theorem is constructed as follows:

- first, assume that p is **true**;
- construct subsequent statements using rules of inference until... q is **true**.

Quantified Direct Proof

Suppose we have a theorem (or lemma/ proposition) of the form: "for all $x \in D$, if $P(x)$ then $Q(x)$", or $\forall x\,(P(x) \rightarrow Q(x))$.
A direct proof of this theorem is constructed as follows:

- pick arbitrary element $c \in D$;

Direct Proofs

Simple Direct Proof

Suppose we have a theorem (or lemma/ proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.
A direct proof of this theorem is constructed as follows:

- first, assume that p is **true**;
- construct subsequent statements using rules of inference until... q is **true**.

Quantified Direct Proof

Suppose we have a theorem (or lemma/ proposition) of the form: "for all $x \in D$, if $P(x)$ then $Q(x)$", or $\forall x\,(P(x) \rightarrow Q(x))$.
A direct proof of this theorem is constructed as follows:

- pick arbitrary element $c \in D$;
- assume that $P(c)$ is **true**, and construct subsequent statements using rules of inference until ... $Q(c)$ is **true**.

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even.

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2\,(-1)$, $-4 = 2\,(-2)$, $0 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 = 2(1010)$. The numbers -3, -7, 1, and 2021 are odd. We have $-3 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 = 2(1010)$. The numbers -3, -7, 1, and 2021 are odd. We have $-3 = 2(-2) + 1$, $-7 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 = 2(1010)$. The numbers -3, -7, 1, and 2021 are odd. We have $-3 = 2(-2) + 1$, $-7 = 2(-4) + 1$, $1 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 = 2(1010)$. The numbers -3, -7, 1, and 2021 are odd. We have $-3 = 2(-2) + 1$, $-7 = 2(-4) + 1$, $1 = (2)(0) + 1$, and
$2021 =$

Mathematical Definition

Mathematical definitions are precise statements that are used to describe mathematical object. Definitions are also facts that are assumed to be true.

Definition

An integer n is even if there is an integer k such that $n = 2k$; and n is odd if there is an integer k such that $n = 2k + 1$.

Example

The numbers -2, -4, 0, and 2020 are even. We have $-2 = 2(-1)$, $-4 = 2(-2)$, $0 = 2(0)$, and $2020 = 2(1010)$. The numbers -3, -7, 1, and 2021 are odd. We have $-3 = 2(-2) + 1$, $-7 = 2(-4) + 1$, $1 = (2)(0) + 1$, and $2021 = 2(1010) + 1$.

Definition

An integer n is called a perfect square if there exists an integer b such that $n = b^2$.

Example

The numbers 4, 9, and 49 are perfect squares, because $4 =$

Definition

An integer n is called a perfect square if there exists an integer b such that $n = b^2$.

Example

The numbers 4, 9, and 49 are perfect squares, because $4 = 2^2$, $9 =$

Definition

An integer n is called a perfect square if there exists an integer b such that $n = b^2$.

Example

The numbers 4, 9, and 49 are perfect squares, because $4 = 2^2$, $9 = 3^2$, and 49

Definition

An integer n is called a perfect square if there exists an integer b such that $n = b^2$.

Example

The numbers 4, 9, and 49 are perfect squares, because $4 = 2^2$, $9 = 3^2$, and $49 = 7^2$.

Definition

An integer n is called a perfect square if there exists an integer b such that $n = b^2$.

Example

The numbers 4, 9, and 49 are perfect squares, because $4 = 2^2$, $9 = 3^2$, and $49 = 7^2$. The numbers 7, 8, and 11 are not perfect squares, because there are no integers a, b, and c such that $7 = a^2$, $8 = b^2$, and $11 = c^2$.

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n =$

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 =$

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 = (2k + 1)^2 =$

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 =$

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 =$

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2\ell + 1$, for some integer $\ell = 2k^2 + 2k$.

Examples of Direct Proofs

Now we are going to prove following theorems.

Theorem

If n is an odd integer, then n^2 is odd.

Proof

1. Let n be an integer.
2. Assume that n is odd, then according to the definition $n = 2k + 1$, for some integer k.
3. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2\ell + 1$, for some integer $\ell = 2k^2 + 2k$.
4. Therefore n^2 is odd. □

(Because n^2 can be expressed as $2(\cdots) + 1$.)

Theorem

Suppose m and n are integers. If m and n are both perfect squares, then mn is also a perfect square.

Proof

Theorem

Suppose m and n are integers. If m and n are both perfect squares, then mn is also a perfect square.

Proof

1. Let m and n be integers.

Theorem

Suppose m and n are integers. If m and n are both perfect squares, then mn is also a perfect square.

Proof

1. Let m and n be integers.
2. Suppose m and n are both perfect square, then $m = b^2$, for some integer b; and $n = c^2$, for some integer c.

Theorem

Suppose m and n are integers. If m and n are both perfect squares, then mn is also a perfect square.

Proof

1. Let m and n be integers.
2. Suppose m and n are both perfect square, then $m = b^2$, for some integer b; and $n = c^2$, for some integer c.
3. We have $mn = b^2c^2 = (bc)^2$, for some integer bc.

Theorem

Suppose m and n are integers. If m and n are both perfect squares, then mn is also a perfect square.

Proof

1. Let m and n be integers.
2. Suppose m and n are both perfect square, then $m = b^2$, for some integer b; and $n = c^2$, for some integer c.
3. We have $mn = b^2c^2 = (bc)^2$, for some integer bc.
4. Therefore mn is also a perfect square. □

(Because mn can be expressed as $(\cdots)^2$.)

Writing a neat and correct proof

Some mathematical proofs are not usually written in separate step-by-step arguments as in our previous example. In many occasions, mathematical proofs are written as narrative arguments that consist of several sentences or paragraphs. Each of these sentences is started with capital letters and ended by a period (.), unless if the beginning of the sentence is a mathematical symbol/notation.

The proofs of our previous theorem can be conveyed as follows:

Proof (If n is odd, then n^2 is also odd.)

The proofs of our previous theorem can be conveyed as follows:

Proof (If n is odd, then n^2 is also odd.)

Let n be an integer. If n is odd, then according to the definition $n = 2k + 1$, for some integer k. We have $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2\ell + 1$, for some integer $\ell = 2k^2 + 2k$. Therefore, n is odd. □

Proof (If m and n are perfect squares, then mn is also a perfect square.)

The proofs of our previous theorem can be conveyed as follows:

Proof (If n is odd, then n^2 is also odd.)

Let n be an integer. If n is odd, then according to the definition $n = 2k + 1$, for some integer k. We have
$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2\ell + 1$, for some integer $\ell = 2k^2 + 2k$. Therefore, n is odd. □

Proof (If m and n are perfect squares, then mn is also a perfect square.)

Let m and n be integers. Suppose m and n are perfect squares, then $m = b^2$, for some integer b; and $n = c^2$, for some integer c. We have $mn = b^2c^2 = (bc)^2$, for some integer bc. Therefore, mn is also a perfect square. □

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2\left(k + \ell\right) + 2 = 2\left(k + \ell + 1\right)$.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2(k + \ell) + 2 = 2(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2(k + \ell) + 2 = 2(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Let a, b, and c be integers.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2(k + \ell) + 2 = 2(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Let a, b, and c be integers. If $a + b$ and $b + c$ are even, then there are integers k and ℓ such that $a + b = 2k$ and $b + c = 2\ell$.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2(k + \ell) + 2 = 2(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Let a, b, and c be integers. If $a + b$ and $b + c$ are even, then there are integers k and ℓ such that $a + b = 2k$ and $b + c = 2\ell$. Observe that $a = 2k - b$ and $c = 2\ell - b$.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2(k + \ell) + 2 = 2(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Let a, b, and c be integers. If $a + b$ and $b + c$ are even, then there are integers k and ℓ such that $a + b = 2k$ and $b + c = 2\ell$. Observe that $a = 2k - b$ and $c = 2\ell - b$. As a result, $a + c = 2k + 2\ell - 2b = 2(k + \ell - b)$.

Exercise 1

Theorem (Theorem 1.1)

The sum of two odd integers is even.

Theorem (Theorem 1.2)

If a, b, c are integers such that: $a + b$ and $b + c$ are both even, then $a + c$ is also even.

Proof (Proof of Theorem 1.1)

Let a and b be integers. If a and b are odd, then there are integers k and ℓ such that $a = 2k + 1$ and $b = 2\ell + 1$. We have $a + b = 2\,(k + \ell) + 2 = 2\,(k + \ell + 1)$. Therefore, $a + b$ is even. □

Proof (Proof of Theorem 1.2)

Let a, b, and c be integers. If $a + b$ and $b + c$ are even, then there are integers k and ℓ such that $a + b = 2k$ and $b + c = 2\ell$. Observe that $a = 2k - b$ and $c = 2\ell - b$. As a result, $a + c = 2k + 2\ell - 2b = 2\,(k + \ell - b)$. Therefore, $a + c$ is even. □

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Let q and r be two rational numbers,

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Let q and r be two rational numbers, then $q = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, d are integers, b and d are nonzero.

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Let q and r be two rational numbers, then $q = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, d are integers, b and d are nonzero. Observe that $q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, with $ad + bc$ and bd are integers.

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Let q and r be two rational numbers, then $q = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, d are integers, b and d are nonzero. Observe that $q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, with $ad + bc$ and bd are integers. Since b and d are nonzero, then $bd \neq 0$.

Exercise 2

Definition

A real number r is rational if there exists integers a and b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number that is not rational is called irrational.

Theorem (Theorem 2)

The sum of two rational numbers is rational.

Proof (Proof of Theorem 2)

Let q and r be two rational numbers, then $q = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, d are integers, b and d are nonzero. Observe that $q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, with $ad + bc$ and bd are integers. Since b and d are nonzero, then $bd \neq 0$. Therefore $q + r$ is rational. $\square$

Contents

Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof (?)

Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof (?)

Since n^2 is odd, then $n^2 = 2k + 1$, for some integer k.

Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof (?)

Since n^2 is odd, then $n^2 = 2k + 1$, for some integer k. We obtain $n = \pm\sqrt{2k+1}$.

Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof (?)

Since n^2 is odd, then $n^2 = 2k + 1$, for some integer k. We obtain $n = \pm\sqrt{2k+1}$.
What's next???

Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof (?)

Since n^2 is odd, then $n^2 = 2k + 1$, for some integer k. We obtain $n = \pm\sqrt{2k+1}$.
What's next???

Direct proof method cannot be used to prove the above theorem.

Indirect Proof by Contraposition

- Suppose we have a theorem (or lemma/proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.

Indirect Proof by Contraposition

- Suppose we have a theorem (or lemma/proposition) in a conditional statement: "if p then q", or $p \to q$.
- Observe that $p \to q$ is equivalent to $\neg q \to \neg p$. Therefore, to prove $p \to q$ is true, we can instead prove that its contrapositive

Indirect Proof by Contraposition

- Suppose we have a theorem (or lemma/proposition) in a conditional statement: "if p then q", or $p \to q$.
- Observe that $p \to q$ is equivalent to $\neg q \to \neg p$. Therefore, to prove $p \to q$ is true, we can instead prove that its contrapositive $\neg q \to \neg p$ is true.

An indirect proof by contraposition of $p \to q$ is equivalent to the direct proof of $\neg q \to \neg p$, which is constructed as follows:

Indirect Proof by Contraposition

- Suppose we have a theorem (or lemma/proposition) in a conditional statement: "if p then q", or $p \rightarrow q$.
- Observe that $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$. Therefore, to prove $p \rightarrow q$ is true, we can instead prove that its contrapositive $\neg q \rightarrow \neg p$ is true.

An indirect proof by contraposition of $p \rightarrow q$ is equivalent to the direct proof of $\neg q \rightarrow \neg p$, which is constructed as follows:

- first, assume that $\neg q$ **is true**;

Indirect Proof by Contraposition

- Suppose we have a theorem (or lemma/proposition) in a conditional statement: "if p then q", or $p \to q$.
- Observe that $p \to q$ is equivalent to $\neg q \to \neg p$. Therefore, to prove $p \to q$ is true, we can instead prove that its contrapositive $\neg q \to \neg p$ is true.

An indirect proof by contraposition of $p \to q$ is equivalent to the direct proof of $\neg q \to \neg p$, which is constructed as follows:

- first, assume that $\neg q$ **is true**;
- construct subsequent statements using rules of inference until ... $\neg p$ **is true**.

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive:

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd.

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to:

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to: if n is even, then n^2 is even.

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to: if n is even, then n^2 is even. Suppose n is even,

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to: if n is even, then n^2 is even. Suppose n is even, then $n = 2k$, for some integer k.

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to: if n is even, then n^2 is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$.

Examples of Indirect Proofs by Contraposition

Theorem

Let n be an integer. If n^2 is odd, then n is odd.

Proof

The statement in the theorem is equivalent to its contrapositive: if n is not odd, then n^2 is not odd. Because for every integer n we have either n is odd or even (but not both), then the contrapositive of the statement is equivalent to: if n is even, then n^2 is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$. Thus, n^2 is even. $\square$

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is:

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n + 2$ is even.

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even,

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n + 2$ is even. Suppose n is even, then $n = 2k$, for some integer k.

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently,
$3n + 2 = 3\,(2k) + 2 = 2\,(3k+1)$.

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $3n+2 = 3(2k)+2 = 2(3k+1)$. Thus, $3n+2$ is even. □

Proof (Proof of Theorem 3.2)

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently,
$3n+2 = 3(2k)+2 = 2(3k+1)$. Thus, $3n+2$ is even. □

Proof (Proof of Theorem 3.2)

The contrapositive of Theorem 3.2 is:

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n + 2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $3n + 2 = 3(2k) + 2 = 2(3k + 1)$. Thus, $3n + 2$ is even. □

Proof (Proof of Theorem 3.2)

The contrapositive of Theorem 3.2 is: if a, b, n are integers such that

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n + 2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n + 2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $3n + 2 = 3(2k) + 2 = 2(3k + 1)$. Thus, $3n + 2$ is even. □

Proof (Proof of Theorem 3.2)

The contrapositive of Theorem 3.2 is: if a, b, n are integers such that $a > \sqrt{n}$ **and** $b > \sqrt{n}$, then $n \neq ab$.

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $3n+2 = 3(2k)+2 = 2(3k+1)$. Thus, $3n+2$ is even. □

Proof (Proof of Theorem 3.2)

The contrapositive of Theorem 3.2 is: if a, b, n are integers such that $a > \sqrt{n}$ **and** $b > \sqrt{n}$, then $n \neq ab$. Suppose a and b are integers such that $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > (\sqrt{n})^2 = n$, which means $ab > n$.

Exercise 3

Theorem (Theorem 3.1)

If n is an integer and $3n+2$ is odd, then n is odd.

Theorem (Theorem 3.2)

If a, b, and n are positive numbers such that $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (Proof of Theorem 3.1)

The contrapositive of Theorem 3.1 is: if n is even, then $3n+2$ is even. Suppose n is even, then $n = 2k$, for some integer k. Consequently, $3n+2 = 3(2k)+2 = 2(3k+1)$. Thus, $3n+2$ is even. □

Proof (Proof of Theorem 3.2)

The contrapositive of Theorem 3.2 is: if a, b, n are integers such that $a > \sqrt{n}$ **and** $b > \sqrt{n}$, then $n \neq ab$. Suppose a and b are integers such that $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > (\sqrt{n})^2 = n$, which means $ab > n$. Therefore, $ab \neq n$. □