Basic Theory of Graph (Part 2)

Isomorphism, Connectivity, Euler and Hamiltonian Path

MZI

School of Computing
Telkom University

SoC Tel-U

May 2023

Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret 2 at Fasilkom UI by Team of Lecturers.
6. Slide for Matematika Diskret. Telkom University, by B. Purnama.

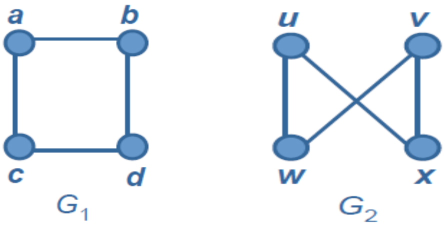
Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

Contents

Graph Isomorphism: Motivation

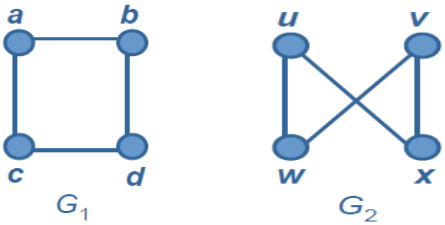
Observe the two following graphs:



Are G_1 and G_2 the same graph?

Graph Isomorphism: Motivation

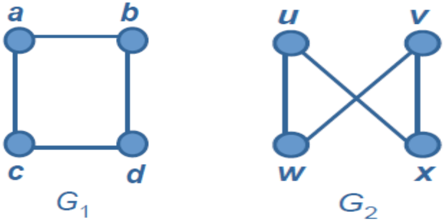
Observe the two following graphs:



Are G_1 and G_2 the same graph? No, because
$V_{G_1} = \{a, b, c, d\} \neq \{u, v, w, x\} = V_{G_2}$.
Can G_2 be redrawn so that G_1 "similar" to G_2?

Graph Isomorphism: Motivation

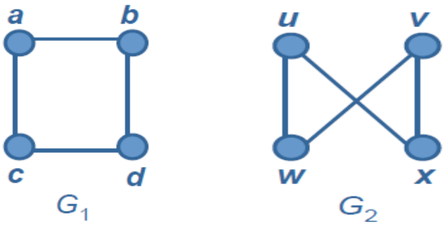
Observe the two following graphs:



Are G_1 and G_2 the same graph? No, because $V_{G_1} = \{a, b, c, d\} \neq \{u, v, w, x\} = V_{G_2}$.
Can G_2 be redrawn so that G_1 "similar" to G_2? Yes.

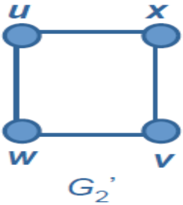
Graph Isomorphism: Motivation

Observe the two following graphs:



Are G_1 and G_2 the same graph? No, because $V_{G_1} = \{a, b, c, d\} \neq \{u, v, w, x\} = V_{G_2}$.

Can G_2 be redrawn so that G_1 "similar" to G_2? Yes.



Graph Isomorphism: Definition

Informally, two graphs G_1 and G_2 are called isomorphic if graph G_2 can be redrawn so that G_2 becomes similar to G_1, or vice versa (G_1 becomes similar to G_2).

Graph Isomorphism: Definition

Informally, two graphs G_1 and G_2 are called isomorphic if graph G_2 can be redrawn so that G_2 becomes similar to G_1, or vice versa (G_1 becomes similar to G_2).

Graph Isomorphism that is discussed in this course involves graphs (directed or undirected) that have no multiple edges.

Definition

Graph Isomorphism: Definition

Informally, two graphs G_1 and G_2 are called isomorphic if graph G_2 can be redrawn so that G_2 becomes similar to G_1, or vice versa (G_1 becomes similar to G_2).
Graph Isomorphism that is discussed in this course involves graphs (directed or undirected) that have no multiple edges.

Definition

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where both of them have no multiple edges, G_1 and G_2 are called isomorphic, written as $G_1 \cong G_2$, if there is an injective total function $f : V_1 \to V_2$ with the properties

$$\{a, b\} \in E_1 \quad \Leftrightarrow \quad \{f(a), f(b)\} \in E_2 \text{ (for undirected graph)}$$
$$(a, b) \in E_1 \quad \Leftrightarrow \quad (f(a), f(b)) \in E_2 \text{ (for directed graph).}$$

The total function f is called as isomorphism.

Graph Isomorphism: Definition

Informally, two graphs G_1 and G_2 are called isomorphic if graph G_2 can be redrawn so that G_2 becomes similar to G_1, or vice versa (G_1 becomes similar to G_2).
Graph Isomorphism that is discussed in this course involves graphs (directed or undirected) that have no multiple edges.

Definition

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where both of them have no multiple edges, G_1 and G_2 are called isomorphic, written as $G_1 \cong G_2$, if there is an injective total function $f : V_1 \to V_2$ with the properties

$$\{a, b\} \in E_1 \quad \Leftrightarrow \quad \{f(a), f(b)\} \in E_2 \text{ (for undirected graph)}$$
$$(a, b) \in E_1 \quad \Leftrightarrow \quad (f(a), f(b)) \in E_2 \text{ (for directed graph).}$$

The total function f is called as isomorphism.

Therefore, two graphs are called isomorphic if there is a one-to-one correspondence between the vertices in the two graphs that preserves the adjacency relationship.

Properties of Two Graphs that are Isomorphic

Theorem

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$ are two isomorphic graphs (with isomorphism f), then

1. $|V_1| = |V_2|$ and $|E_1| = |E_2|$,
2. for every $a \in V_1$ we have $\deg(a) = \deg(f(a))$.

That means two isomorphic graphs G_1 and G_2 have following characteristics:

Properties of Two Graphs that are Isomorphic

Theorem

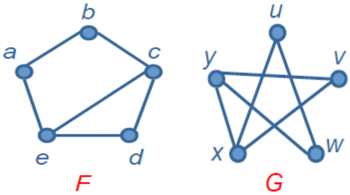
Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$ are two isomorphic graphs (with isomorphism f), then

1. $|V_1| = |V_2|$ and $|E_1| = |E_2|$,
2. for every $a \in V_1$ we have $\deg(a) = \deg(f(a))$.

That means two isomorphic graphs G_1 and G_2 have following characteristics:

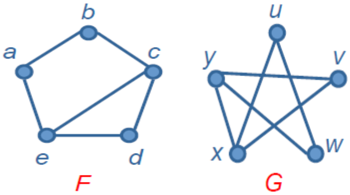
1. The number of vertices in G_1 is identical to the number of vertices in G_2.
2. The number of edges in G_1 is identical to the number of edges in G_2.
3. The degree of each vertex that corresponds to each other in the two graphs is identical.

Identifying Isomorphic Graph



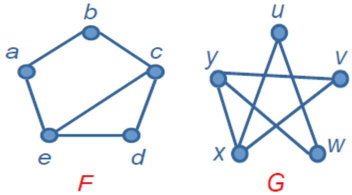
Are F and G isomorphic?

Identifying Isomorphic Graph



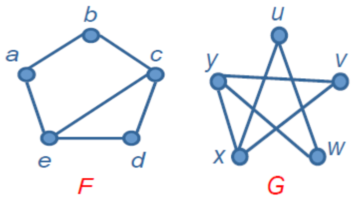
Are F and G isomorphic? Observe that $|V_F| = |V_G| =$

Identifying Isomorphic Graph



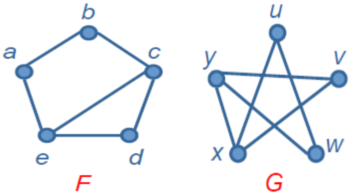
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| =$

Identifying Isomorphic Graph



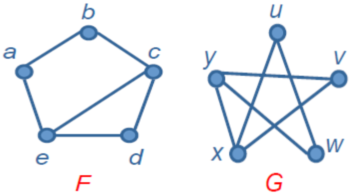
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$.

Identifying Isomorphic Graph



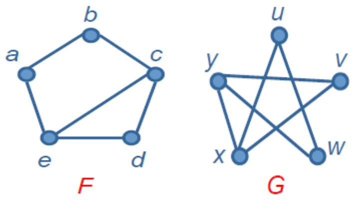
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic.

Identifying Isomorphic Graph



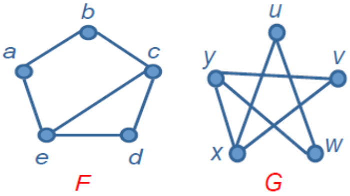
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism:

Identifying Isomorphic Graph



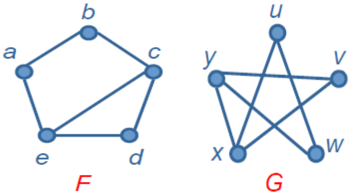
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$.

Identifying Isomorphic Graph



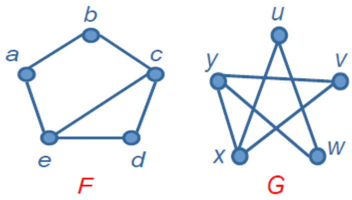
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow$

Identifying Isomorphic Graph



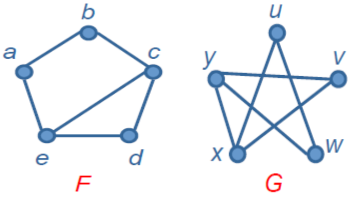
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \rightarrow V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,

Identifying Isomorphic Graph



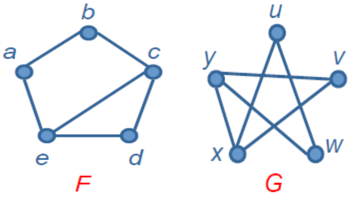
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \rightarrow V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$, $\{b, c\} \in E_F \Leftrightarrow$

Identifying Isomorphic Graph



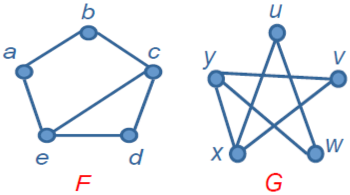
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$, $\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,

Identifying Isomorphic Graph



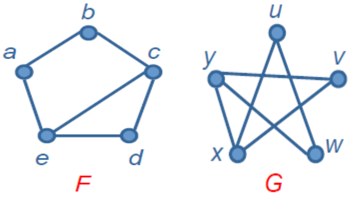
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow$

Identifying Isomorphic Graph



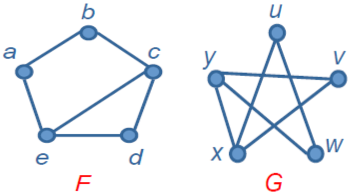
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,

Identifying Isomorphic Graph



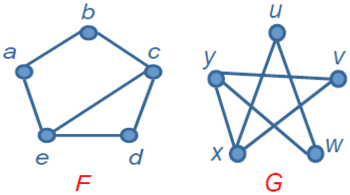
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \rightarrow V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow$

Identifying Isomorphic Graph



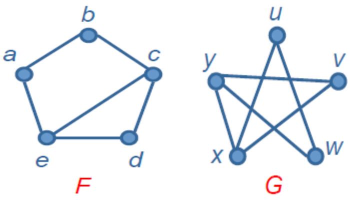
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow \{f(d), f(e)\} = \{v, x\} \in E_G$,

Identifying Isomorphic Graph



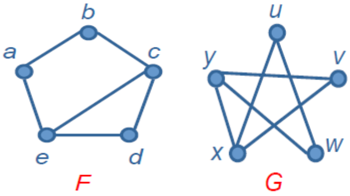
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow \{f(d), f(e)\} = \{v, x\} \in E_G$,
$\{e, c\} \in E_F \Leftrightarrow$

Identifying Isomorphic Graph



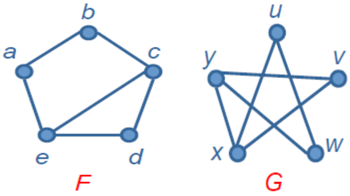
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow \{f(d), f(e)\} = \{v, x\} \in E_G$,
$\{e, c\} \in E_F \Leftrightarrow \{f(e), f(c)\} = \{x, y\} \in E_G$,

Identifying Isomorphic Graph

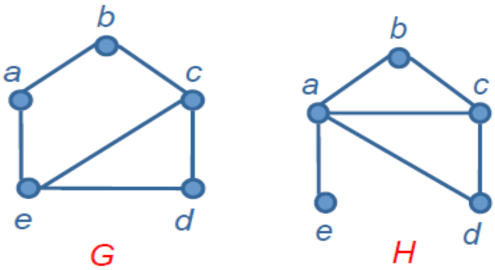


Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow \{f(d), f(e)\} = \{v, x\} \in E_G$,
$\{e, c\} \in E_F \Leftrightarrow \{f(e), f(c)\} = \{x, y\} \in E_G$,
$\{e, a\} \in E_F \Leftrightarrow$

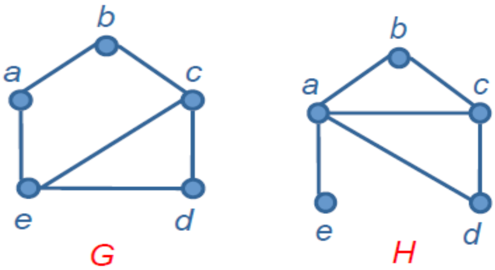
Identifying Isomorphic Graph



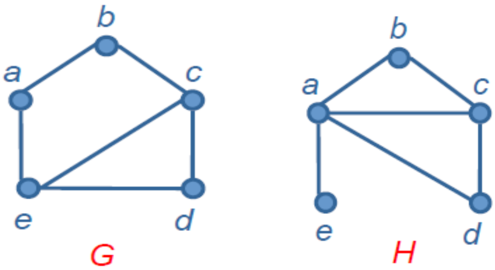
Are F and G isomorphic? Observe that $|V_F| = |V_G| = 5$ and $|E_F| = |E_G| = 6$. So F and G are potentially isomorphic. Then notice that the mapping $f : V_F \to V_G$ is an isomorphism: $f(a) = u$, $f(b) = w$, $f(c) = y$, $f(d) = v$, $f(e) = x$. We have: $\{a, b\} \in E_F \Leftrightarrow \{f(a), f(b)\} = \{u, w\} \in E_G$,
$\{b, c\} \in E_F \Leftrightarrow \{f(b), f(c)\} = \{w, y\} \in E_G$,
$\{c, d\} \in E_F \Leftrightarrow \{f(c), f(d)\} = \{y, v\} \in E_G$,
$\{d, e\} \in E_F \Leftrightarrow \{f(d), f(e)\} = \{v, x\} \in E_G$,
$\{e, c\} \in E_F \Leftrightarrow \{f(e), f(c)\} = \{x, y\} \in E_G$,
$\{e, a\} \in E_F \Leftrightarrow \{f(e), f(a)\} = \{x, u\} \in E_G$.



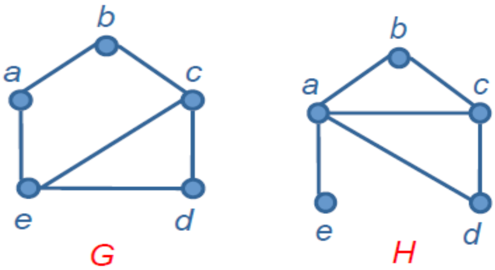
Are G and H isomorphic?



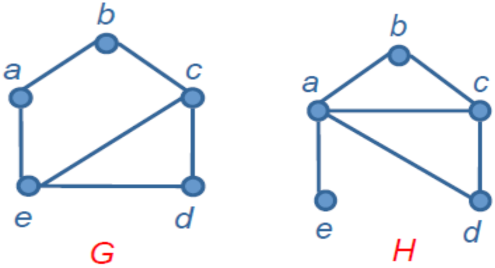
Are G and H isomorphic? Observe that $|V_G| = |V_H| =$



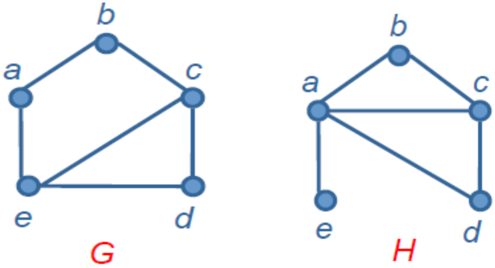
Are G and H isomorphic? Observe that $|V_G| = |V_H| = 5$ and $|E_G| = |E_H| =$



Are G and H isomorphic? Observe that $|V_G| = |V_H| = 5$ and $|E_G| = |E_H| = 6$.

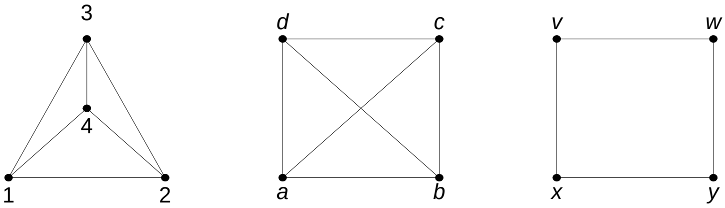


Are G and H isomorphic? Observe that $|V_G| = |V_H| = 5$ and $|E_G| = |E_H| = 6$. However, notice that in H there is vertex e of degree 1, while in G there is no vertex of degree 1. Moreover, in H there is vertex a of degree 4 while in G there is no vertex of degree 4.



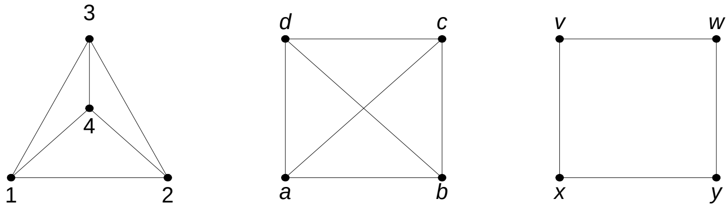
Are G and H isomorphic? Observe that $|V_G| = |V_H| = 5$ and $|E_G| = |E_H| = 6$. However, notice that in H there is vertex e of degree 1, while in G there is no vertex of degree 1. Moreover, in H there is vertex a of degree 4 while in G there is no vertex of degree 4. Therefore $G \ncong H$.

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



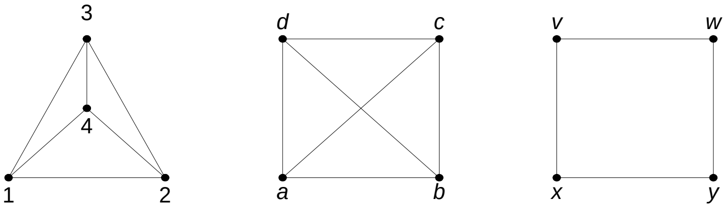
We have

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



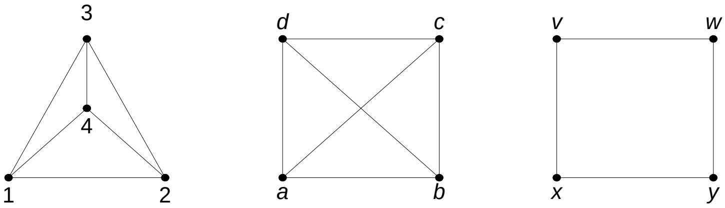
We have $|V_1| = |V_2| = |V_3| = 4$,

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



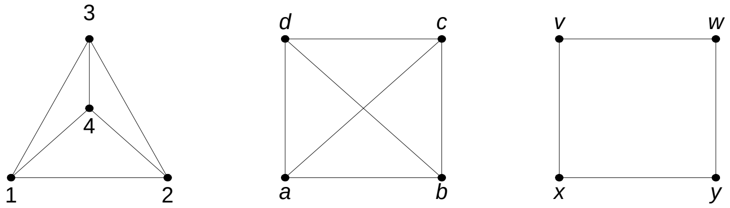
We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$,

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



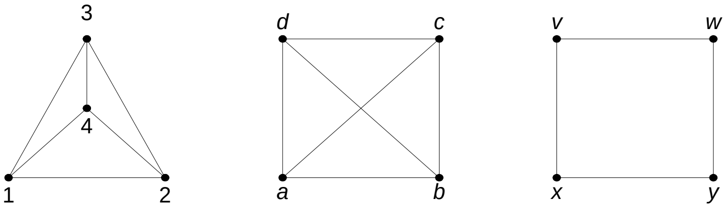
We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) =$

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



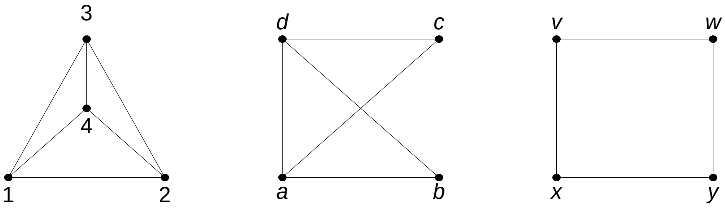
We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) =$

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



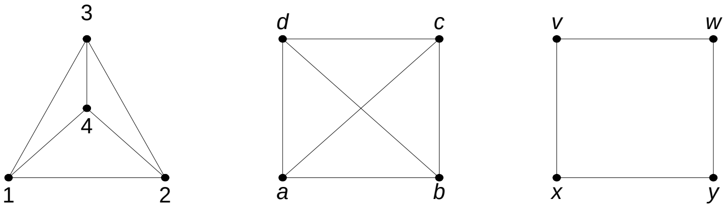
We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) =$

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



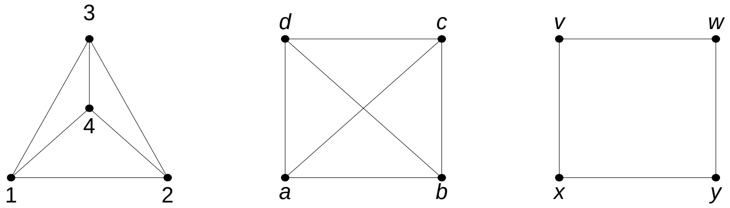
We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) =$

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$. We have

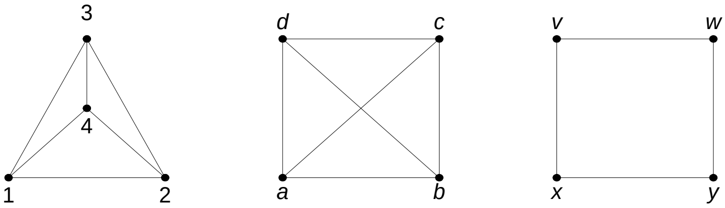
Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$. We have

- $\{1,2\} \in E_1 \Leftrightarrow \{a,b\} \in E_2$, $\{1,3\} \in E_1 \Leftrightarrow \{a,c\} \in E_2$, $\{1,4\} \in E_1 \Leftrightarrow \{a,d\} \in E_2$.

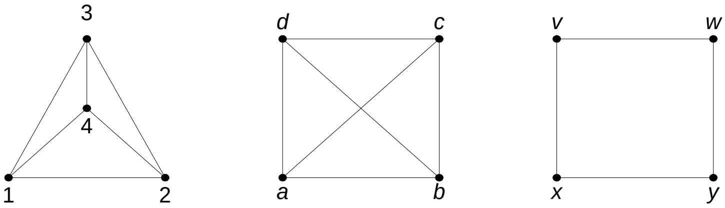
Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.



We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$. We have

- $\{1, 2\} \in E_1 \Leftrightarrow \{a, b\} \in E_2$, $\{1, 3\} \in E_1 \Leftrightarrow \{a, c\} \in E_2$, $\{1, 4\} \in E_1 \Leftrightarrow \{a, d\} \in E_2$.
- $\{2, 3\} \in E_1 \Leftrightarrow \{b, c\} \in E_2$, $\{2, 4\} \in E_1 \Leftrightarrow \{b, d\} \in E_2$.

Suppose $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, $G_3 = (V_3, E_3)$ (from left to right) are the following graphs respectively.

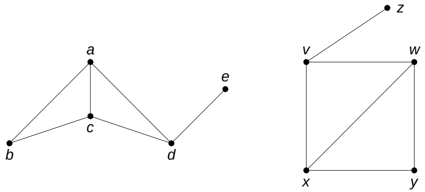


We have $|V_1| = |V_2| = |V_3| = 4$, $|E_1| = |E_2| = 6$, $|E_3| = 4$, so G_1 are potentially isomorphic with G_2, but it is clear that $G_1 \ncong G_3$ and $G_2 \ncong G_3$. Then notice that the following mapping $f : V_{E_1} \to V_{E_2}$ is an isomorphism: $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$. We have

- $\{1,2\} \in E_1 \Leftrightarrow \{a,b\} \in E_2$, $\{1,3\} \in E_1 \Leftrightarrow \{a,c\} \in E_2$, $\{1,4\} \in E_1 \Leftrightarrow \{a,d\} \in E_2$.
- $\{2,3\} \in E_1 \Leftrightarrow \{b,c\} \in E_2$, $\{2,4\} \in E_1 \Leftrightarrow \{b,d\} \in E_2$.
- $\{3,4\} \in E_1 \Leftrightarrow \{c,d\} \in E_2$.

Identifying Isomorphic Graphs via Adjacency Matrix

Suppose G and H are the two following graphs (from left to right respectively).



To check whether $G \cong H$, we can form an adjacency matrix for each graph, namely $\mathbf{A}_G$ and $\mathbf{A}_H$, and see whether the row and column of $\mathbf{A}_H$ can be permuted so that $\mathbf{A}_H = \mathbf{A}_G$.

We have

$$\mathbf{A}_G =$$

We have

$$\mathbf{A}_G = \begin{array}{c|ccccc|} & a & b & c & d & e \\ a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 0 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

and $\mathbf{A}_H =$

We have

$$\mathbf{A}_G = \begin{array}{c|ccccc|} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 0 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

and $$\mathbf{A}_H = \begin{array}{c|ccccc|} & v & w & x & y & z \\ \hline v & 0 & 1 & 1 & 0 & 1 \\ w & 1 & 0 & 1 & 1 & 0 \\ x & 1 & 1 & 0 & 1 & 0 \\ y & 0 & 1 & 1 & 0 & 0 \\ z & 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \mathbf{A}_H =$$

We have

$$\mathbf{A}_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

and $\mathbf{A}_H = \begin{array}{c} \\ v \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} v & w & x & y & z \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}, \mathbf{A}_H = \begin{array}{c} \\ x \\ y \\ w \\ v \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & w & v & z \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array},$

observe that the function

We have

$$\mathbf{A}_G = \begin{array}{c|ccccc|} & a & b & c & d & e \\ a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 0 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 0 \end{array}$$

and $\mathbf{A}_H = \begin{array}{c|ccccc|} & v & w & x & y & z \\ v & 0 & 1 & 1 & 0 & 1 \\ w & 1 & 0 & 1 & 1 & 0 \\ x & 1 & 1 & 0 & 1 & 0 \\ y & 0 & 1 & 1 & 0 & 0 \\ z & 1 & 0 & 0 & 0 & 0 \end{array}$, $\mathbf{A}_H = \begin{array}{c|ccccc|} & x & y & w & v & z \\ x & 0 & 1 & 1 & 1 & 0 \\ y & 1 & 0 & 1 & 0 & 0 \\ w & 1 & 1 & 0 & 1 & 0 \\ v & 1 & 0 & 1 & 0 & 1 \\ z & 0 & 0 & 0 & 1 & 0 \end{array}$,

observe that the function $f(a) = x$, $f(b) = y$, $f(c) = w$, $f(d) = v$, and $f(e) = z$ is an isomorphism, hence $G \cong H$.

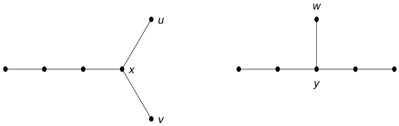
More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

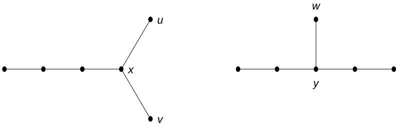
However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).



More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).

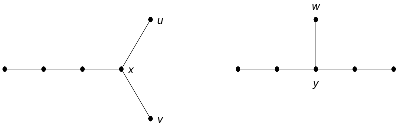


Although $|V_1| = |V_2|$ and $|E_1| = |E_2|$, we have $G_1 \ncong G_2$.

More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).

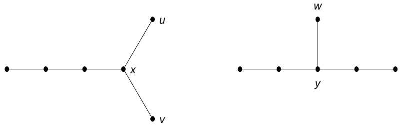


Although $|V_1| = |V_2|$ and $|E_1| = |E_2|$, we have $G_1 \ncong G_2$. Suppose $G_1 \cong G_2$, then the only possible isomorphism must make $f(x) =$

More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).

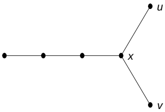


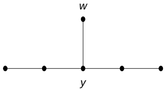
Although $|V_1| = |V_2|$ and $|E_1| = |E_2|$, we have $G_1 \ncong G_2$. Suppose $G_1 \cong G_2$, then the only possible isomorphism must make $f(x) = y$.

More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).



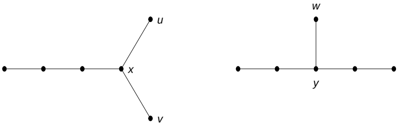


Although $|V_1| = |V_2|$ and $|E_1| = |E_2|$, we have $G_1 \ncong G_2$. Suppose $G_1 \cong G_2$, then the only possible isomorphism must make $f(x) = y$. In G_1, vertex x is adjacent to two pendant vertices, namely u and v.

More about Graph Isomorphism

We have already seen that graphs G_1 and G_2 that are isomorphic have following characteristics: the number of vertices in G_1 is identical to the number of vertices in G_2, the number of edges in G_1 is identical to the number of edges in G_2, and the degree of each vertex that corresponds to each other on the two graphs are identical.

However, sometimes the above characteristics are not enough and we need to draw G_1 and G_2 to verify the isomorphism visually. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs as follows (from left to right respectively).



Although $|V_1| = |V_2|$ and $|E_1| = |E_2|$, we have $G_1 \ncong G_2$. Suppose $G_1 \cong G_2$, then the only possible isomorphism must make $f(x) = y$. In G_1, vertex x is adjacent to two pendant vertices, namely u and v. Meanwhile, in G_2, vertex y only adjacent with one pendant vertex.

Graph Isomorphism Problem

Graph isomorphism problem is the following computational problem.

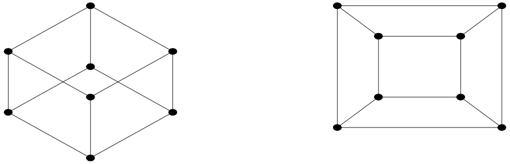
Graph Isomorphism Problem

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, check whether $G_1 \cong G_2$.

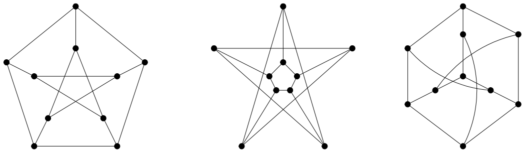
Not all of graph isomorphism problems can be solved easily. Furthermore, until now, there is no efficient algorithm to solve this problem. Manual verification of graph isomorphism needs meticulousness and specific insights.

Some Examples of Isomorphic Graphs

The two following graphs are isomorphic graphs.



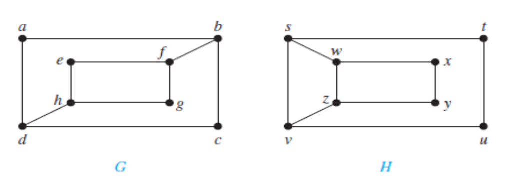
These following three graphs are isomorphic graphs.



Exercise 1: Graph Isomorphism

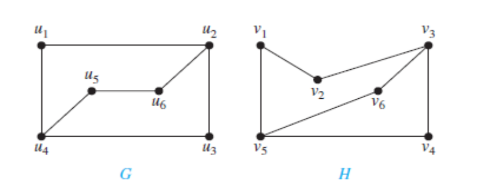
Exercise

1. Check whether the following graphs G and H are isomorphic.



Graph G and H

2. Check whether the following graphs G and H are isomorphic.

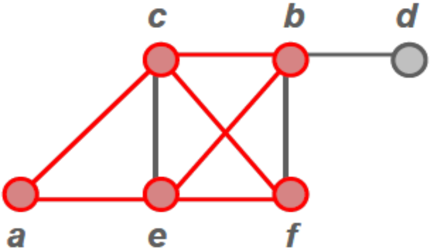


Graph G and H

Contents

Connectivity: Motivation

Notice the following graph:



1. Can we go from vertex a to all other vertices?
2. Is there any "path" from vertex a to vertex b that passes through all of the edges in the graph exactly once except the edge $\{b, d\}$?
3. Can we visit all vertices and back to the initial vertex where each vertex is visited only once?

Path Definition

Definition (Path Definition)

Given an undirected graph $G = (V, E, f)$ and an integer $n \geq 0$, a path of length n from vertex u to v is a sequence of n edges

$$e_1, e_2, \ldots, e_n, \text{ where}$$

$$f(e_1) = \{t_0, t_1\}, \; f(e_2) = \{t_1, t_2\}, \; \ldots, \; f(e_n) = \{t_{n-1}, t_n\}, \; t_0 = u \text{ and } t_n = v.$$

When G is a simple graph (no multiple edges neither loop), then a path of length n as explained before can be written as $t_0, t_1, \ldots, t_n$. Usually, this path is written as $\langle t_0, t_1, \ldots, t_n \rangle$.

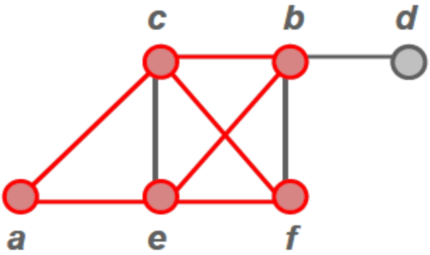
Definition

A path is called pass through vertices $x_1, x_2, \ldots, x_{n-1}$ or traverse the edges $e_1, e_2, \ldots, e_n$.

Path definition for directed graphs is analogous to the above definition.

Example of Path

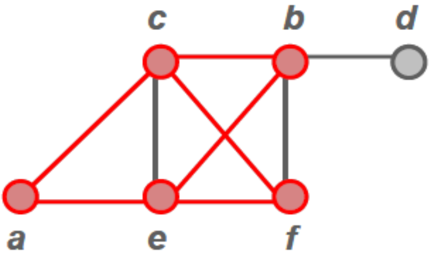
Notice the following graph:



In the above graph: $\langle a, e, f, c \rangle$ is a path of length

Example of Path

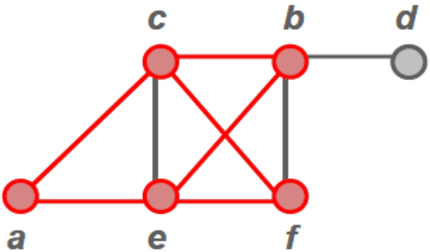
Notice the following graph:



In the above graph: $\langle a, e, f, c \rangle$ is a path of length 3, $\langle a, e, b, c, a \rangle$ is a path of length

Example of Path

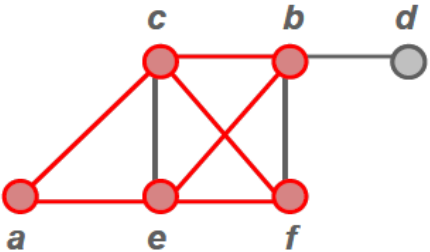
Notice the following graph:



In the above graph: $\langle a, e, f, c \rangle$ is a path of length 3, $\langle a, e, b, c, a \rangle$ is a path of length 4, $\langle a, e, f, e, f \rangle$ is a path of length

Example of Path

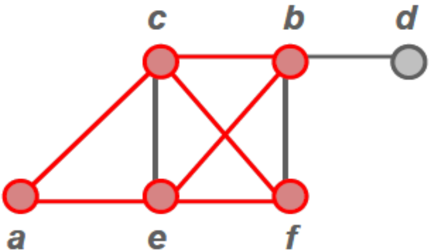
Notice the following graph:



In the above graph: $\langle a, e, f, c \rangle$ is a path of length 3, $\langle a, e, b, c, a \rangle$ is a path of length 4, $\langle a, e, f, e, f \rangle$ is a path of length 4, and $\langle a, e, c, a, e, a \rangle$ is a path of length

Example of Path

Notice the following graph:



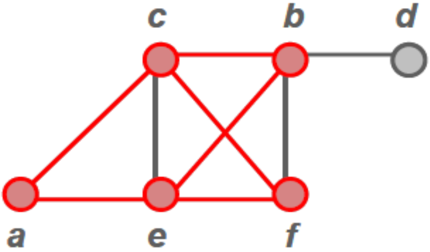
In the above graph: $\langle a, e, f, c \rangle$ is a path of length 3, $\langle a, e, b, c, a \rangle$ is a path of length 4, $\langle a, e, f, e, f \rangle$ is a path of length 4, and $\langle a, e, c, a, e, a \rangle$ is a path of length 5.

Definition and Example of Circuit

Definition (Definition of Circuit or Cycle)

A path $\langle t_0, t_1, \ldots, t_n \rangle$ is called a circuit or cycle if $t_0 = t_n$ and its length **is not zero**.

Notice the following graph:



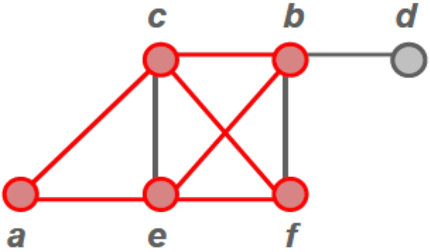
In the above graph: $\langle e, f, b, c, e \rangle$ is a circuit of length

Definition and Example of Circuit

Definition (Definition of Circuit or Cycle)

A path $\langle t_0, t_1, \ldots, t_n \rangle$ is called a circuit or cycle if $t_0 = t_n$ and its length **is not zero**.

Notice the following graph:



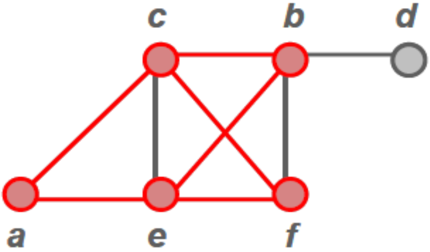
In the above graph: $\langle e, f, b, c, e \rangle$ is a circuit of length 4, $\langle a, e, f, c, a \rangle$ is a circuit of length

Definition and Example of Circuit

Definition (Definition of Circuit or Cycle)

A path $\langle t_0, t_1, \ldots, t_n \rangle$ is called a circuit or cycle if $t_0 = t_n$ and its length **is not zero**.

Notice the following graph:



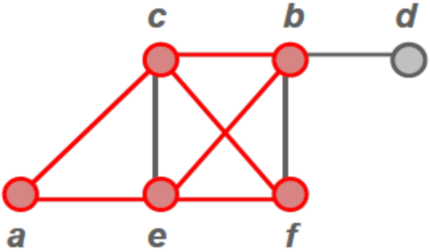
In the above graph: $\langle e, f, b, c, e \rangle$ is a circuit of length 4, $\langle a, e, f, c, a \rangle$ is a circuit of length 4, $\langle a, e, c, a, e, a \rangle$ is a circuit of length

Definition and Example of Circuit

Definition (Definition of Circuit or Cycle)

A path $\langle t_0, t_1, \ldots, t_n \rangle$ is called a circuit or cycle if $t_0 = t_n$ and its length **is not zero**.

Notice the following graph:



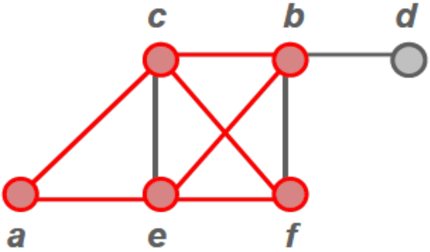
In the above graph: $\langle e, f, b, c, e \rangle$ is a circuit of length 4, $\langle a, e, f, c, a \rangle$ is a circuit of length 4, $\langle a, e, c, a, e, a \rangle$ is a circuit of length 5, and $\langle d, b, e, f, b, d \rangle$ is a circuit of length

Definition and Example of Circuit

Definition (Definition of Circuit or Cycle)

A path $\langle t_0, t_1, \ldots, t_n \rangle$ is called a circuit or cycle if $t_0 = t_n$ and its length **is not zero**.

Notice the following graph:



In the above graph: $\langle e, f, b, c, e \rangle$ is a circuit of length 4, $\langle a, e, f, c, a \rangle$ is a circuit of length 4, $\langle a, e, c, a, e, a \rangle$ is a circuit of length 5, and $\langle d, b, e, f, b, d \rangle$ is a circuit of length 5.

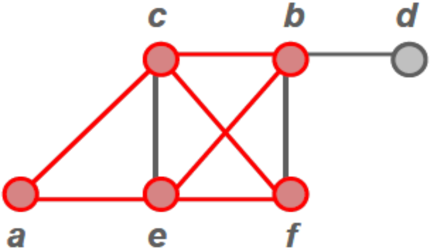
Simple and Elementary Circuit

Definition (Definition of Simple and Elementary Path/Circuit)

A path (or circuit) is called simple if the path (or circuit) **has no** (or not pass through) same edges more than once. Furthermore, a path (or circuit) is called elementary if it **has no** (or not pass through) same vertex more than once.

Example of Simple and Elementary Circuit

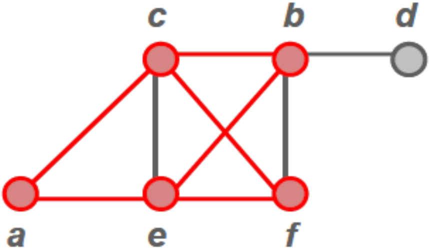
Notice the following graph:



In the above graph: $\langle a, e, a, e, a \rangle$

Example of Simple and Elementary Circuit

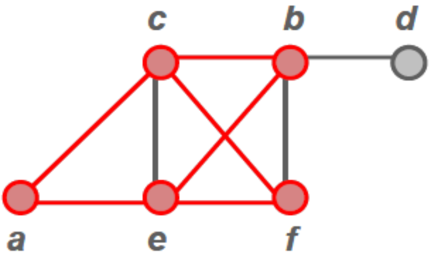
Notice the following graph:



In the above graph: $\langle a, e, a, e, a \rangle$ is not a simple path and not an elementary path, $\langle a, e, b, f, e \rangle$

Example of Simple and Elementary Circuit

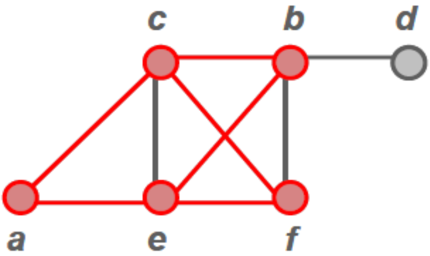
Notice the following graph:



In the above graph: $\langle a, e, a, e, a \rangle$ is not a simple path and not an elementary path, $\langle a, e, b, f, e \rangle$ is a simple path but not an elementary path, and $\langle a, e, f, b, d \rangle$

Example of Simple and Elementary Circuit

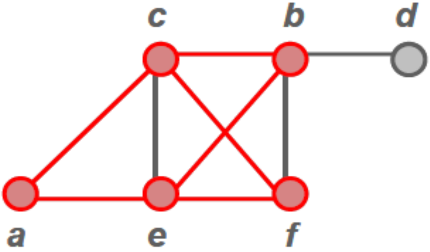
Notice the following graph:



In the above graph: $\langle a, e, a, e, a \rangle$ is not a simple path and not an elementary path, $\langle a, e, b, f, e \rangle$ is a simple path but not an elementary path, and $\langle a, e, f, b, d \rangle$ is a simple path as well as an elementary path.

Example of Simple and Elementary Circuit

Notice the following graph:



In the above graph: $\langle a, e, a, e, a \rangle$ is not a simple path and not an elementary path, $\langle a, e, b, f, e \rangle$ is a simple path but not an elementary path, and $\langle a, e, f, b, d \rangle$ is a simple path as well as an elementary path. Is there any elementary path that is not simple on the above graph?

Connectedness Definition

Definition (Connectedness for Undirected Graphs)

Suppose $G = (V, E, f)$ is an undirected graph. Two vertices u and v are called connected if there is a path from u to v. Afterwards, G is called connected if there is a path from u to v **for every** $u, v \in V$ where $u \neq v$.

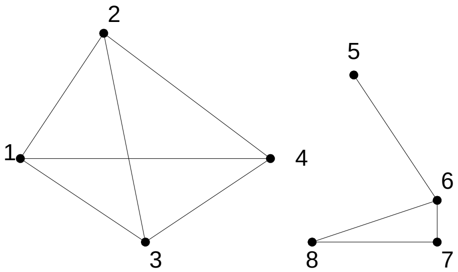
Definition (Connectedness for Directed Graphs)

Suppose $G = (V, E, f)$ is a directed graph. Vertex u is called connected to v (or vertex v is connected from u) if there is a path from u to v. Then

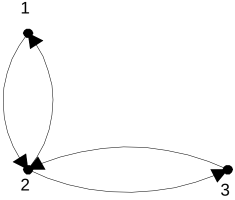
1. G is called strongly connected if there is a path from u to v **and** from v to u for every $u, v \in V$ where $u \neq v$,
2. G is called weakly connected if the undirected graph G_0 that is obtained from graph G by eliminating its direction is a connected graph.

If G is not a connected graph, then G is called as a disconnected graph.

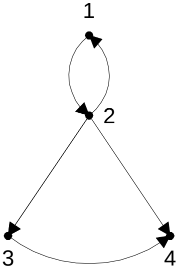
The example of a disconnected graph is as follows.



The example of strongly connected directed graph is as follows.



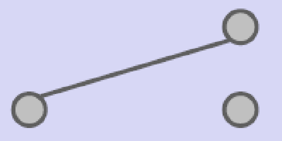
The example of weakly connected directed graph is as follows.



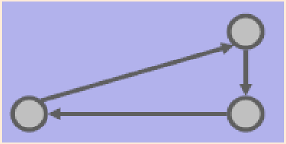
Exercise 2: Connectivity

Exercise

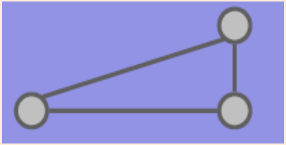
Classify the following graphs based on their connectivity!



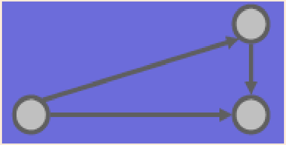
Graph G_1



Graph G_2



Graph G_3



Graph G_4

Connected Component

Definition (Connected Component on Undirected Graphs)

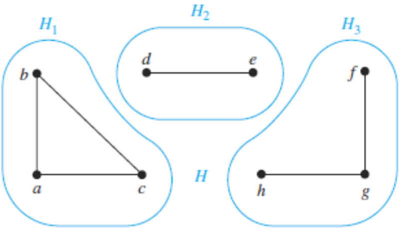
Connected component of a graph G is a subgraph of G that is connected and it is not a proper subgraph of other connected subgraph.

Connected Component

Definition (Connected Component on Undirected Graphs)

Connected component of a graph G is a subgraph of G that is connected and it is not a proper subgraph of other connected subgraph.

Notice the following illustration.



The above graph is a graph H that contains three connected components, namely H_1, H_2, and H_3.

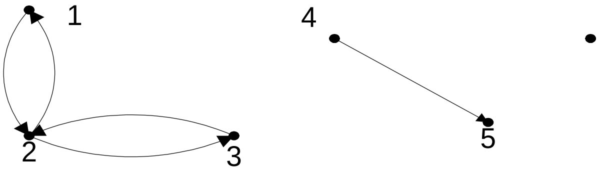
Definition (Strongly Connected Component on Directed Graphs)

A strongly connected component of a graph G is a subgraph of G that is strongly connected and it is not a proper subgraph of other connected subgraph.

Definition (Strongly Connected Component on Directed Graphs)

A strongly connected component of a graph G is a subgraph of G that is strongly connected and it is not a proper subgraph of other connected subgraph.

For example, the following graph G has two strongly connected components (the leftmost subgraph and the rightmost subgraph).



Cut Set

Definition (Cut Set of Connected Graph)

Suppose $G = (V, E)$ is a connected undirected graph, the set of edges $C \subseteq E$ is called a cut set if

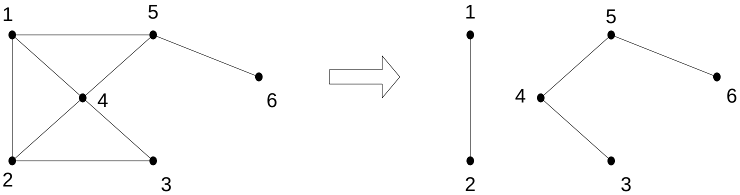
1. eliminating edges in C causes G become disconnected,
2. there is no $D \subset C$ that can cause G become disconnected by eliminating the edges on D.

Intuitively, a cut set **cannot** contain another cut set as its proper subset.

Notice the following illustration:

Set $C = \{\{1,5\},\{1,4\},\{2,3\},\{2,4\}\}$ is a cut set. Some of the other cut sets are:

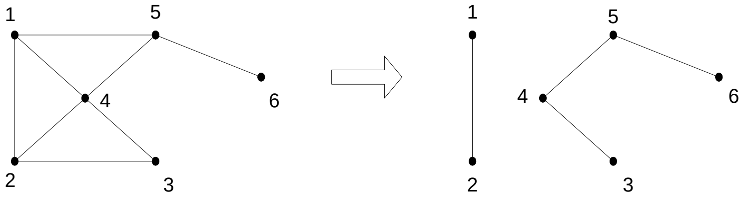
Notice the following illustration:



Set $C = \{\{1,5\},\{1,4\},\{2,3\},\{2,4\}\}$ is a cut set. Some of the other cut sets are:

1. $C = \{\{1,5\},\{4,5\}\}$, notice that all of proper subset of C is not a cut set;

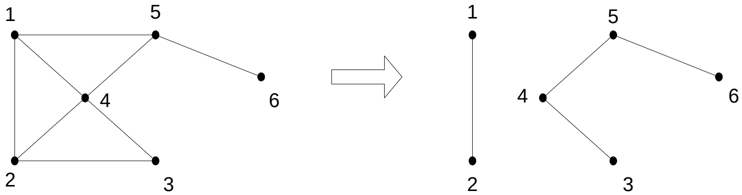
Notice the following illustration:



Set $C = \{\{1,5\},\{1,4\},\{2,3\},\{2,4\}\}$ is a cut set. Some of the other cut sets are:

1. $C = \{\{1,5\},\{4,5\}\}$, notice that all of proper subset of C is not a cut set;
2. $C = \{\{1,2\},\{1,4\},\{1,5\}\}$, notice that all of proper subset of C is not a cut set;

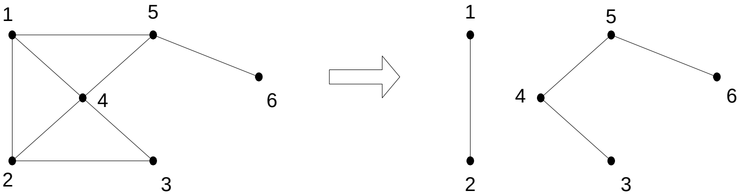
Notice the following illustration:



Set $C = \{\{1,5\},\{1,4\},\{2,3\},\{2,4\}\}$ is a cut set. Some of the other cut sets are:

1. $C = \{\{1,5\},\{4,5\}\}$, notice that all of proper subset of C is not a cut set;
2. $C = \{\{1,2\},\{1,4\},\{1,5\}\}$, notice that all of proper subset of C is not a cut set;
3. $C = \{\{5,6\}\}$, notice that all of proper subset of C is not a cut set.

Notice the following illustration:



Set $C = \{\{1,5\},\{1,4\},\{2,3\},\{2,4\}\}$ is a cut set. Some of the other cut sets are:

1. $C = \{\{1,5\},\{4,5\}\}$, notice that all of proper subset of C is not a cut set;
2. $C = \{\{1,2\},\{1,4\},\{1,5\}\}$, notice that all of proper subset of C is not a cut set;
3. $C = \{\{5,6\}\}$, notice that all of proper subset of C is not a cut set.

Set $\{\{1,5\},\{4,5\},\{3,4\}\}$ **is not a cut set** because $\{\{1,5\},\{4,5\}\}$ is already a cut set.

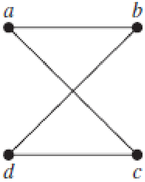
Counting the Number of Paths Between Two Vertices

The number of paths of particular length between two vertices in a graph can be obtained through its adjacency matrix.

Theorem

Suppose $G = (V, E, f)$ is a graph (either directed or undirected, may have multiple edges or loop) where $V = \{v_1, v_2, \ldots, v_n\}$ with adjacency matrix $\mathbf{A}_G$. If $\mathbf{A}_G[i, j]$ is the number of edges from v_i to v_j, then the number of different paths of length r $(r = 1, 2, \ldots)$ from v_i to v_j is equal to $[i, j]$-th entry of the matrix $\mathbf{A}_G^r$.

Suppose G is the following graph.

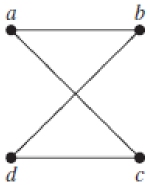


Graph G

Suppose we want to know the number of paths in G with length of 4 from c to b. In the graph G, with the vertices order a, b, c, d, we have

$$\mathbf{A}_G =$$

Suppose G is the following graph.

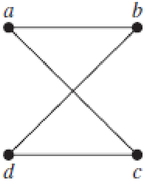


Graph G

Suppose we want to know the number of paths in G with length of 4 from c to b. In the graph G, with the vertices order a, b, c, d, we have

$$\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{A}_G^4 =$$

Suppose G is the following graph.



Graph G

Suppose we want to know the number of paths in G with length of 4 from c to b. In the graph G, with the vertices order a, b, c, d, we have

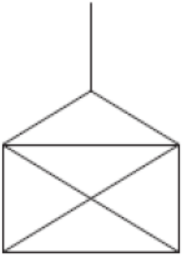
$$\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{A}_G^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

So there are 8 paths with length of 4 from c to b.

Contents

Motivation: Euler Path and Circuit

Can the pattern in the following picture be drawn using a pencil with continuous movement (without lifting the pencil) and every line is drawn only once?



Definition of Euler Path and Circuit

Definition (Multigraph)

An undirected graph $G = (V, E, f)$ is called a multigraph if G may have multiple edges but has no loop.

Definition (Euler Path and Circuit)

Suppose $G = (V, E, f)$ is a multigraph or directed graph that has no loop, an Euler path is a simple path that contain every edge in G. Then, an Euler circuit is an Euler path that starts and ends on the same vertex .

Therefore, an Euler path of a graph is a path that traverses every edge on the graph **exactly once**. In addition an Euler circuit of a graph is a circuit that traverses every edge on the graph **exactly once**.

Eulerian Graph and Semi-Eulerian Graph

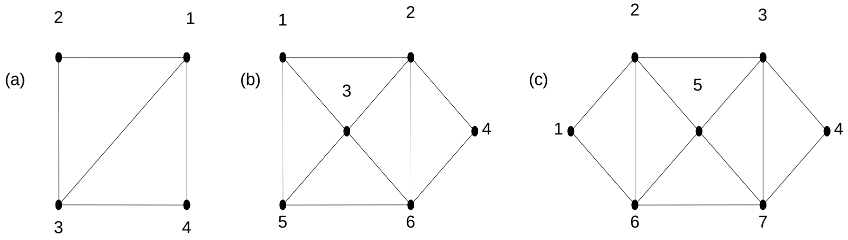
Definition (Eulerian Graph and Semi-Eulerian Graph)

A graph that has an Euler circuit is called an Eulerian graph. If the graph has no Euler circuit but has Euler path, then the graph is called as a semi-Eulerian graph.

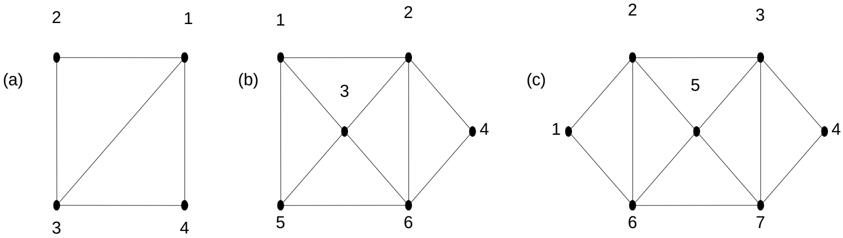
Remark

Every graph that has an Euler circuit obviously has an Euler path, but not vice versa.

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.

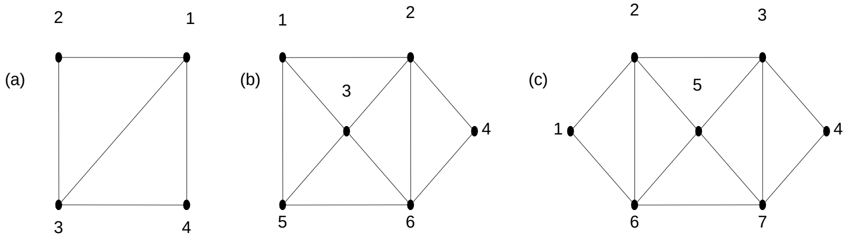


Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



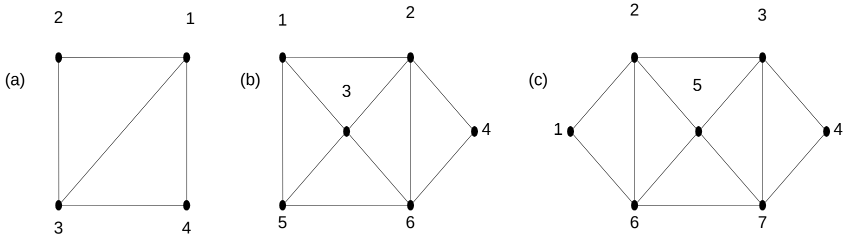
1. Graph G_1 has an Euler path, one of them is

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



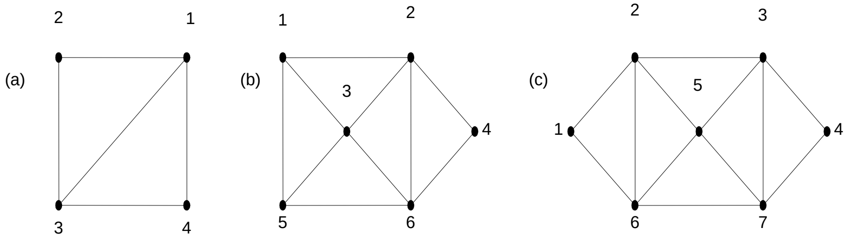
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$.

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



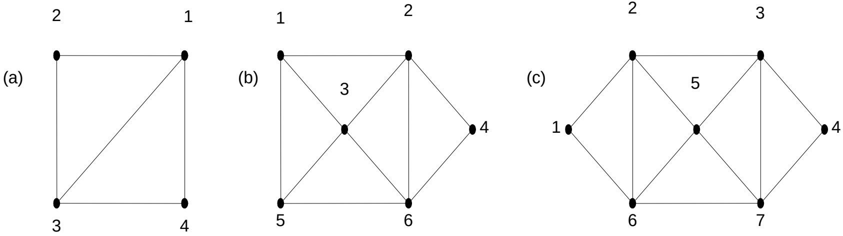
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



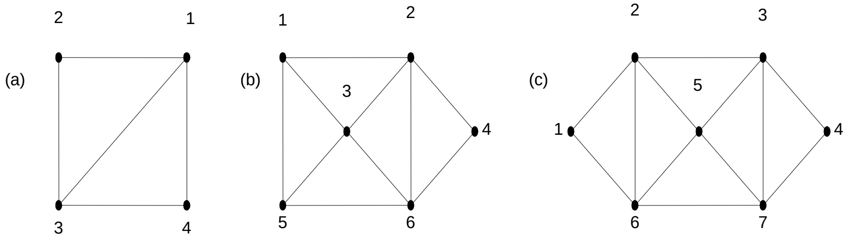
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).
2. Graph G_2 has an Euler path, one of them is

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



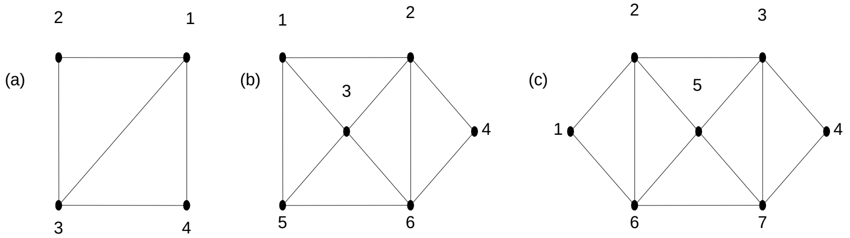
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).
2. Graph G_2 has an Euler path, one of them is $\langle 1, 2, 4, 6, 2, 3, 6, 5, 1, 3, 5 \rangle$.

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



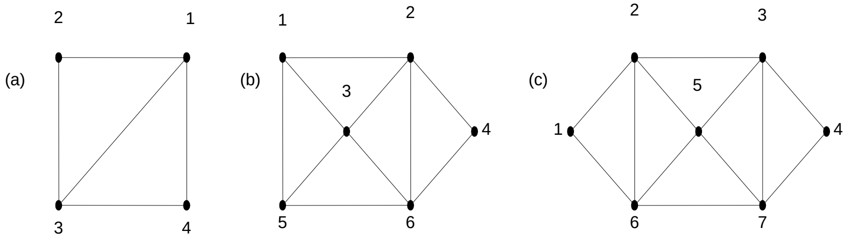
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).
2. Graph G_2 has an Euler path, one of them is $\langle 1, 2, 4, 6, 2, 3, 6, 5, 1, 3, 5 \rangle$. However, G_2 has no Euler circuit (check it).

Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).
2. Graph G_2 has an Euler path, one of them is $\langle 1, 2, 4, 6, 2, 3, 6, 5, 1, 3, 5 \rangle$. However, G_2 has no Euler circuit (check it).
3. Graph G_3 has an Euler circuit, one of them is

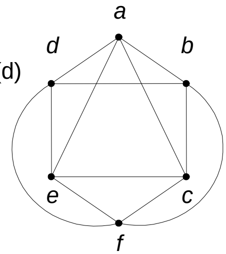
Suppose G_1, G_2, and G_3 are the following graphs (a), (b), and (c) respectively.



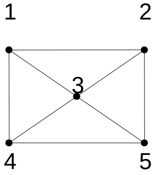
1. Graph G_1 has an Euler path, one of them is $\langle 1, 2, 3, 1, 4, 3 \rangle$. However, G_1 has no Euler circuit (check it).
2. Graph G_2 has an Euler path, one of them is $\langle 1, 2, 4, 6, 2, 3, 6, 5, 1, 3, 5 \rangle$. However, G_2 has no Euler circuit (check it).
3. Graph G_3 has an Euler circuit, one of them is $\langle 1, 2, 3, 4, 7, 3, 5, 7, 6, 5, 2, 6, 1 \rangle$.

Suppose G_4, G_5, and G_6 are the following graphs (d), (e), and (f) respectively.

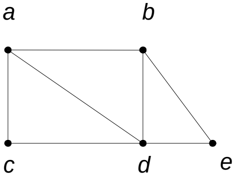
(d)



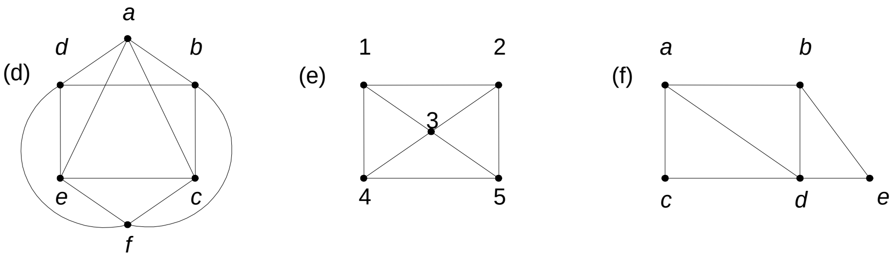
(e)



(f)

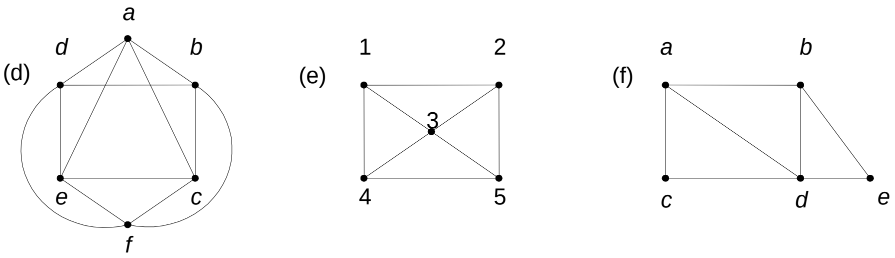


Suppose G_4, G_5, and G_6 are the following graphs (d), (e), and (f) respectively.



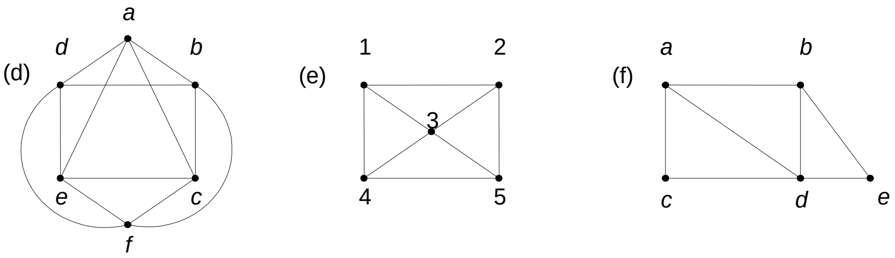
1. Graph G_4 has an Euler circuit, one of them is $\langle a, c, f, e, c, b, d, e, a, d, f, b, a \rangle$.

Suppose G_4, G_5, and G_6 are the following graphs (d), (e), and (f) respectively.



1. Graph G_4 has an Euler circuit, one of them is $\langle a, c, f, e, c, b, d, e, a, d, f, b, a \rangle$.
2. Graph G_5 has no Euler circuit and path.

Suppose G_4, G_5, and G_6 are the following graphs (d), (e), and (f) respectively.



1. Graph G_4 has an Euler circuit, one of them is $\langle a, c, f, e, c, b, d, e, a, d, f, b, a \rangle$.
2. Graph G_5 has no Euler circuit and path.
3. Graph G_6 has an Euler path, one of them is $\langle a, b, e, d, c, a, d, b \rangle$.

Theorem about Euler Path and Circuit for Undirected Graphs

Theorem

A multigraph $G = (V, E, f)$ has an Euler circuit if and only if G is connected and every vertex has an even degree.

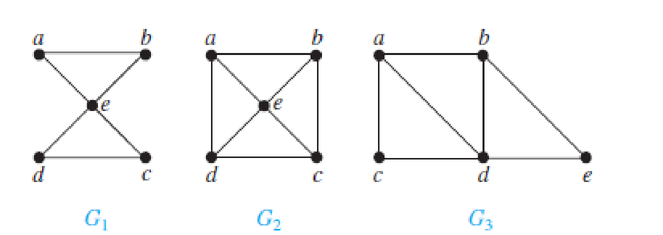
Theorem

A multigraph $G = (V, E, f)$ has an Euler path but has no Euler circuit if and only if G is connected and it has **exactly two vertices** of odd degree.

Exercise 3: Euler Path & Circuit of Undirected Graphs

Exercise

Check whether the following graphs have Eulerian circuit? If no, then check whether the graph has Euler path.



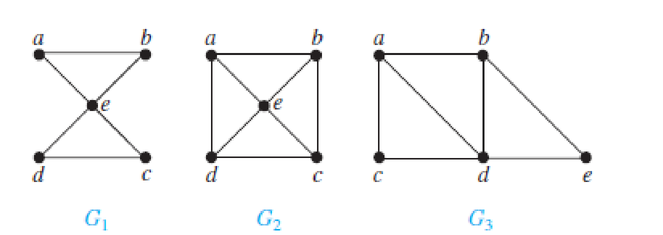
Solution:

- Graph G_1

Exercise 3: Euler Path & Circuit of Undirected Graphs

Exercise

Check whether the following graphs have Eulerian circuit? If no, then check whether the graph has Euler path.



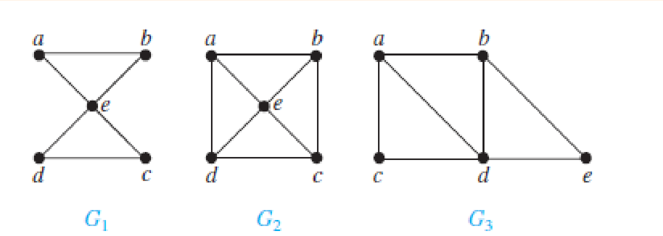
Solution:

- Graph G_1 has an Euler circuit, because the degree of each vertex is even, one of the circuit is $\langle a, b, e, c, d, e, a\rangle$.
- Graph G_2

Exercise 3: Euler Path & Circuit of Undirected Graphs

Exercise

Check whether the following graphs have Eulerian circuit? If no, then check whether the graph has Euler path.



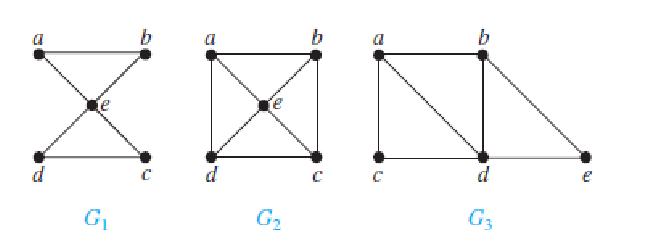
Solution:

- Graph G_1 has an Euler circuit, because the degree of each vertex is even, one of the circuit is $\langle a, b, e, c, d, e, a\rangle$.
- Graph G_2 has no Euler path, because there are four vertices with odd degree, namely $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 3$.
- Graph G_3

Exercise 3: Euler Path & Circuit of Undirected Graphs

Exercise

Check whether the following graphs have Eulerian circuit? If no, then check whether the graph has Euler path.



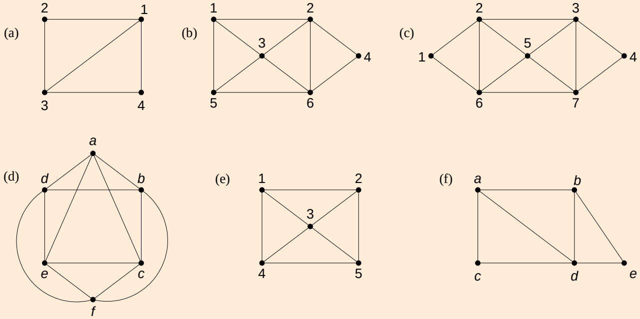
Solution:

- Graph G_1 has an Euler circuit, because the degree of each vertex is even, one of the circuit is $\langle a, b, e, c, d, e, a \rangle$.
- Graph G_2 has no Euler path, because there are four vertices with odd degree, namely $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 3$.
- Graph G_3 has no Euler circuit but it has Euler path, one of the path is $\langle a, b, e, d, c, a, d, b \rangle$.

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

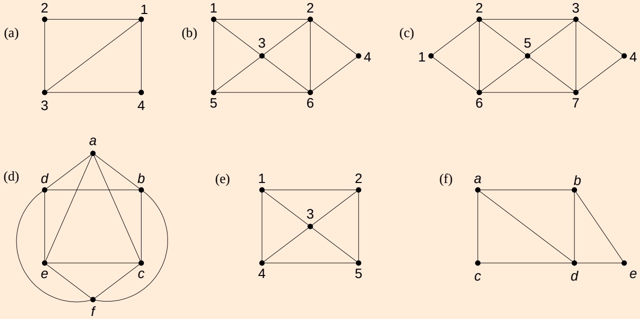


Solution:

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

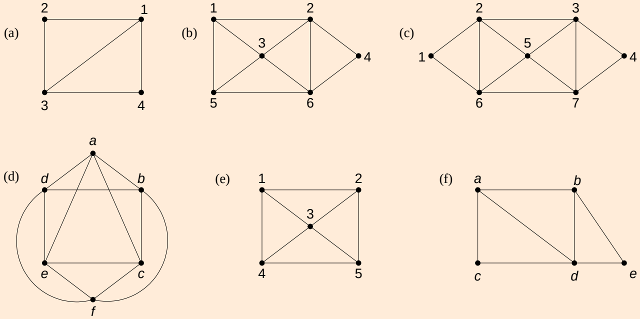


Solution: a) semi-Eulerian graph,

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

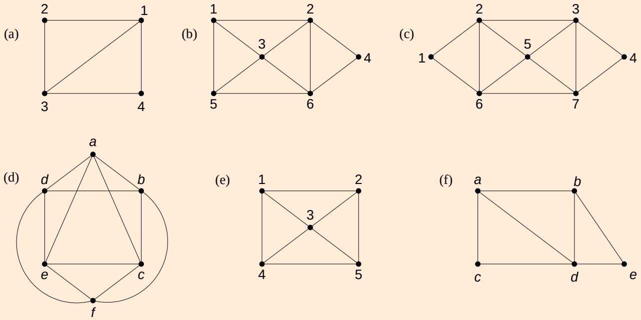


Solution: a) semi-Eulerian graph, b) semi-Eulerian graph,

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

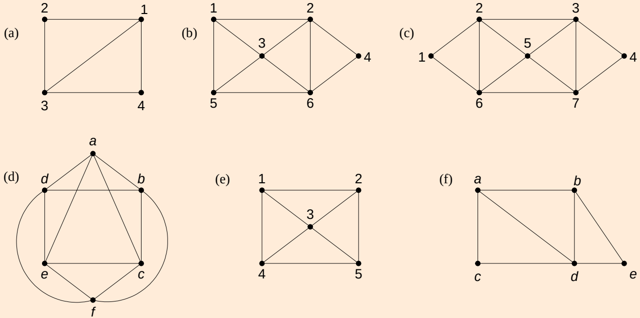


Solution: a) semi-Eulerian graph, b) semi-Eulerian graph, c) Eulerian graph,

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

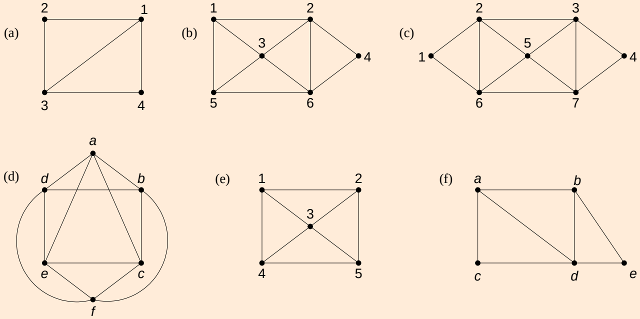


Solution: a) semi-Eulerian graph, b) semi-Eulerian graph, c) Eulerian graph, d) Eulerian graph,

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.

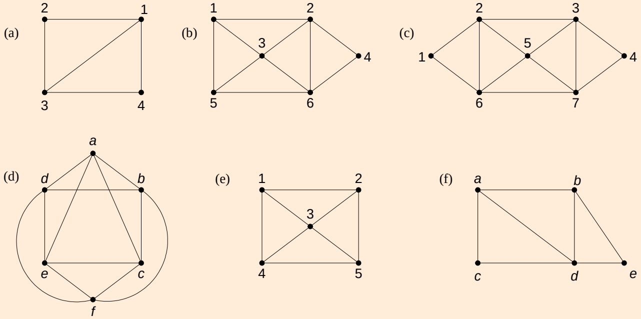


Solution: a) semi-Eulerian graph, b) semi-Eulerian graph, c) Eulerian graph, d) Eulerian graph, e) neither,

Exercise 4: Eulerian and Semi-Eulerian Graph

Exercise

Check whether the following graphs are Eulerian graph, semi-Eulerian graph, or neither.



Solution: a) semi-Eulerian graph, b) semi-Eulerian graph, c) Eulerian graph, d) Eulerian graph, e) neither, f) semi-Eulerian graph.

Theorem about Euler Path and Circuit for Directed Graph

Theorem

A directed graph $G = (V, E)$ has an Euler circuit if and only if G is connected and every vertex has identical in-degree and out-degree, in other words $\deg_{in}(v) = \deg_{out}(v)$ or $\deg^{-}(v) = \deg^{+}(v)$ for every $v \in V$.

Theorem

A directed graph $G = (V, E)$ has an Euler path but **has no** Euler circuit if and only if G is connected and every vertex has identical in-degree and out-degree, except for two vertices a and b with the following properties:

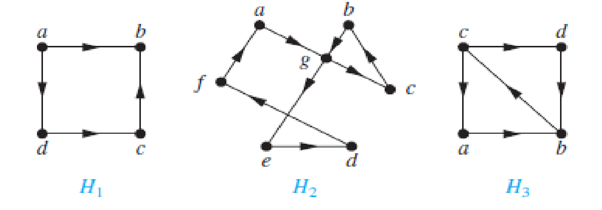
1. $\deg_{out}(a) = \deg_{in}(a) + 1$, or $\deg^{+}(a) = \deg^{-}(a) + 1$
2. $\deg_{in}(b) = \deg_{out}(b) + 1$, or $\deg^{-}(b) = \deg^{+}(b) + 1$.

This means there are exactly two vertices, the first vertex has the out-degree one greater than the in-degree, the second vertex has the in-degree one greater than the out-degree.

Exercise 5: Euler Path & Circuit of Directed Graphs

Exercise

Check whether the following graphs have Euler circuit? If not, check whether the graph has Euler path.

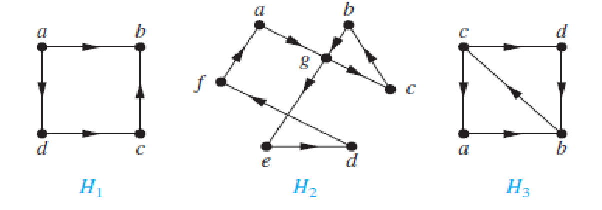


Solution:

Exercise 5: Euler Path & Circuit of Directed Graphs

Exercise

Check whether the following graphs have Euler circuit? If not, check whether the graph has Euler path.



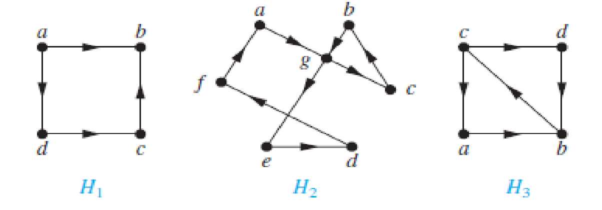
Solution:

1. Graph H_1 has no Euler path, because $\deg^+(a) = 2$ but $\deg^-(a) = 0$.

Exercise 5: Euler Path & Circuit of Directed Graphs

Exercise

Check whether the following graphs have Euler circuit? If not, check whether the graph has Euler path.



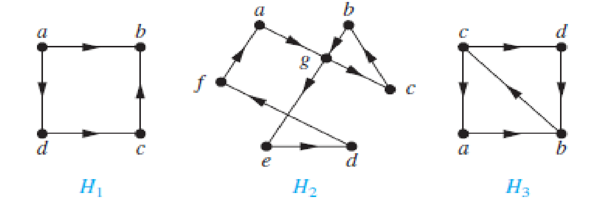
Solution:

1. Graph H_1 has no Euler path, because $\deg^+(a) = 2$ but $\deg^-(a) = 0$.
2. Graph H_2 has an Euler circuit, one of the circuit is $\langle a, g, c, b, g, e, d, f, a \rangle$.

Exercise 5: Euler Path & Circuit of Directed Graphs

Exercise

Check whether the following graphs have Euler circuit? If not, check whether the graph has Euler path.



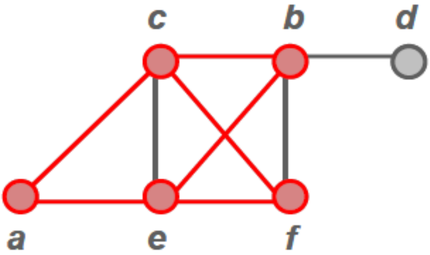
Solution:

1. Graph H_1 has no Euler path, because $\deg^+(a) = 2$ but $\deg^-(a) = 0$.
2. Graph H_2 has an Euler circuit, one of the circuit is $\langle a, g, c, b, g, e, d, f, a\rangle$.
3. Graph H_3 has no Euler circuit but it has an Euler path, one of the path is $\langle c, a, b, c, d, b\rangle$.

Contents

Motivation: Hamilton Path and Circuit

Observe the following graph:



Graph G

Is there any circuit that traverses all vertices in the graph G exactly once? If it isn't, then is there any path that traverses all the vertices in the graph G exactly once?

Definition of Hamilton Path and Circuit

In this course, Hamilton path and circuit are considered on simple graphs (graphs with no multiple edges nor loop).

Definition

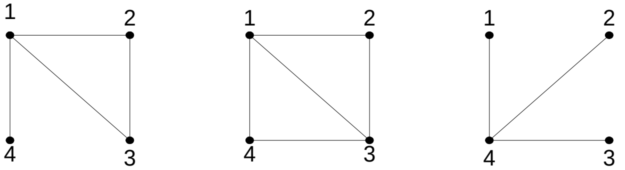
Suppose $G = (V, E)$ is a simple graph. A Hamilton path is a simple path that traverses every vertex on G exactly once. A Hamilton circuit is a simple circuit that traverses every vertex on G exactly once, except for initial vertex (which is identical to the terminal vertex) that is traversed exactly twice .

Definition

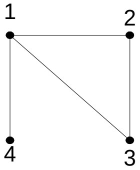
A graph that has a Hamilton circuit is called Hamiltonian graph, while a graph that only has a Hamilton path (but does not have Hamilton circuit) is called semi-Hamiltonian graph.

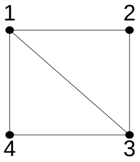
Notice that if $G = (V, E)$ is a semi-Hamilton graph and $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path on G then $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$.

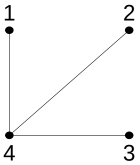
Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).



Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).

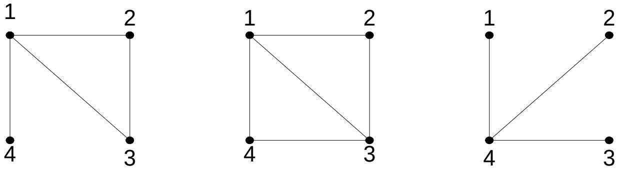






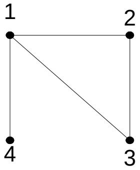
1. Graph G_1 has a Hamilton path, one of them is $\langle 4, 1, 3, 2 \rangle$.

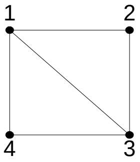
Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).

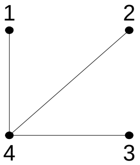


1. Graph G_1 has a Hamilton path, one of them is $\langle 4, 1, 3, 2 \rangle$. However, it is not possible for G_1 to have a Hamilton circuit. This is because every circuit that contain vertex 4 must contain vertex 1 more than once.

Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).

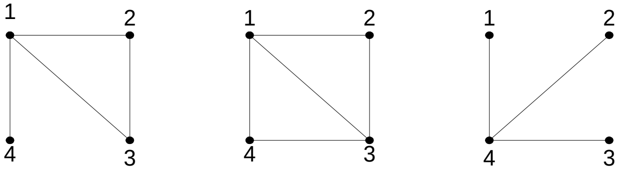






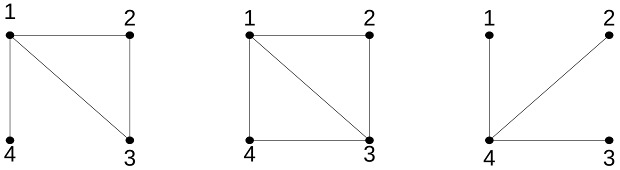
1. Graph G_1 has a Hamilton path, one of them is $\langle 4, 1, 3, 2 \rangle$. However, it is not possible for G_1 to have a Hamilton circuit. This is because every circuit that contain vertex 4 must contain vertex 1 more than once.
2. Graph G_2 has a Hamilton circuit, one of them is $\langle 1, 2, 3, 4, 1 \rangle$.

Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).



1. Graph G_1 has a Hamilton path, one of them is $\langle 4, 1, 3, 2 \rangle$. However, it is not possible for G_1 to have a Hamilton circuit. This is because every circuit that contain vertex 4 must contain vertex 1 more than once.
2. Graph G_2 has a Hamilton circuit, one of them is $\langle 1, 2, 3, 4, 1 \rangle$.
3. Graph G_3 has no Hamilton path.

Suppose G_1, G_2, and G_3 are respectively the following graphs (from left to right).

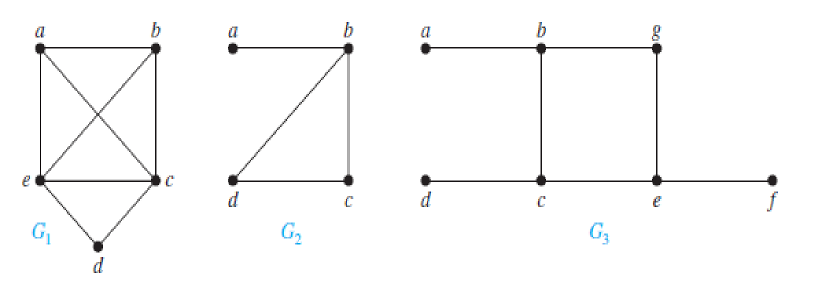


1. Graph G_1 has a Hamilton path, one of them is $\langle 4, 1, 3, 2 \rangle$. However, it is not possible for G_1 to have a Hamilton circuit. This is because every circuit that contain vertex 4 must contain vertex 1 more than once.
2. Graph G_2 has a Hamilton circuit, one of them is $\langle 1, 2, 3, 4, 1 \rangle$.
3. Graph G_3 has no Hamilton path. This is because every path that traverses all vertices in G_3 and contain vertex 1 as well as vertex 3 must contain vertex 4 more than once.

Exercise 6: Hamilton Path and Circuit

Exercise

Check whether the following graphs have Hamilton circuit! If they're not, check whether the graph has Hamilton path!



Graph G_1, G_2, and G_3

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely $\langle a, b, c, d, e, a \rangle$.

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely $\langle a, b, c, d, e, a \rangle$.
2. G_2 has no Hamilton circuit, but G_2 has a Hamilton path, namely

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely $\langle a, b, c, d, e, a \rangle$.
2. G_2 has no Hamilton circuit, but G_2 has a Hamilton path, namely $\langle a, b, c, d \rangle$.

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely $\langle a, b, c, d, e, a \rangle$.
2. G_2 has no Hamilton circuit, but G_2 has a Hamilton path, namely $\langle a, b, c, d \rangle$. G_2 has no Hamilton circuit because every circuit that traverses all vertices in G_2 must contain the edge $\{a, b\}$ at least twice.

Solution of Exercise 6

1. G_1 has a Hamilton circuit, namely $\langle a, b, c, d, e, a \rangle$.
2. G_2 has no Hamilton circuit, but G_2 has a Hamilton path, namely $\langle a, b, c, d \rangle$. G_2 has no Hamilton circuit because every circuit that traverses all vertices in G_2 must contain the edge $\{a, b\}$ at least twice.
3. G_3 has no Hamilton path. This is because every path that contains all vertices in G_3 must contain one of the edges $\{a, b\}$, $\{e, f\}$, or $\{c, d\}$ more than once.

Theorems about Hamilton Path and Circuit

Theorem

A complete graph K_n for $n \geq 3$ has Hamilton circuit.

Theorem (Dirac's Theorem)

If $G = (V, E)$ is a simple connected graph with $|V| \geq 3$ that satisfies

$$\deg(v) \geq \frac{|V|}{2} \text{ for every } v \in V,$$

then G has Hamilton circuit.

Theorem (Ore's Theorem)

If $G = (V, E)$ is a simple connected graph with $|V| \geq 3$ that satisfies

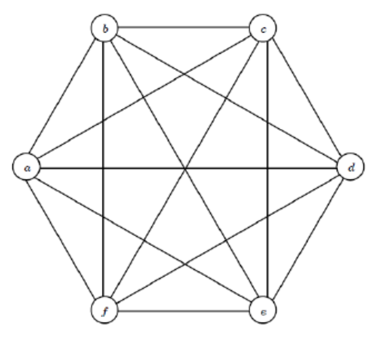
$$\deg(u) + \deg(v) \geq |V|, \text{ for every } u, v \in V \text{ that are not adjacent},$$

then G has Hamilton circuit.

Exercise 7: Counting The Number of Different Hamilton Circuits

Exercise

Determine the number of different Hamilton circuits with initial and terminal vertex a on the following graph.



We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),
3. there are 3 choices for v_3 (because $v_3 \neq v_2 \neq v_1 \neq a$),

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),
3. there are 3 choices for v_3 (because $v_3 \neq v_2 \neq v_1 \neq a$),
4. there are 2 choices for v_4 (because $v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$), and

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),
3. there are 3 choices for v_3 (because $v_3 \neq v_2 \neq v_1 \neq a$),
4. there are 2 choices for v_4 (because $v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$), and
5. there is 1 choice for v_5 (because $v_5 \neq v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$).

Therefore, there are

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),
3. there are 3 choices for v_3 (because $v_3 \neq v_2 \neq v_1 \neq a$),
4. there are 2 choices for v_4 (because $v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$), and
5. there is 1 choice for v_5 (because $v_5 \neq v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$).

Therefore, there are $5! = 120$ possible circuits.

We consider the graph K_6 with set of vertices $V = \{a, b, c, d, e, f\}$. Any Hamilton circuit with initial and terminal vertex a on the graph must have the following form

$$\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle ,$$

where $v_i \in V$ for every $1 \leq i \leq 5$. Because we consider the graph K_6, then every vertex is adjacent to each other, therefore:

1. there are 5 choices for v_1 (because $v_1 \neq a$),
2. there are 4 choices for v_2 (because $v_2 \neq v_1 \neq a$),
3. there are 3 choices for v_3 (because $v_3 \neq v_2 \neq v_1 \neq a$),
4. there are 2 choices for v_4 (because $v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$), and
5. there is 1 choice for v_5 (because $v_5 \neq v_4 \neq v_3 \neq v_2 \neq v_1 \neq a$).

Therefore, there are $5! = 120$ possible circuits. However, because the graph is undirected, then the circuit

$$\langle a, v_5, v_4, v_3, v_2, v_1, a \rangle$$

is considered to be similar with $\langle a, v_1, v_2, v_3, v_4, v_5, a \rangle$, therefore, there is only $\frac{120}{2} = 60$ different Hamilton circuits.

The Number of Hamilton Circuit on K_n

Theorem

There are $\frac{(n-1)!}{2}$ different Hamilton circuits in the complete graph K_n.

Theorem

There are $\frac{n-1}{2}$ disjoint Hamilton circuits (the set of edges on the circuit is disjoint) in a complete graph K_n where $n \geq 3$ and n is an odd number.

Theorem

There are $\frac{n-2}{2}$ disjoint Hamilton circuits (the set of edges on the circuit is disjoint) in a complete graph K_n where $n \geq 4$ and n is an even number.