Basic Theory of Graph (Part 1)

Some Formal Definitions of Graph – Matrix Representation of Graph

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret 2 at Fasilkom UI by Team of Lecturers.
6. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

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Background

Graph is an important object in Discrete Math and has many implementations, one of them is in topology design of communication networks.

We can use graph to **model the connectedness** between discrete objects. One of them is a graph that describe connectedness between cities in Central Java (*here, we view connectedness from the availability of the road connecting the cities*).

Background

Graph is an important object in Discrete Math and has many implementations, one of them is in topology design of communication networks.

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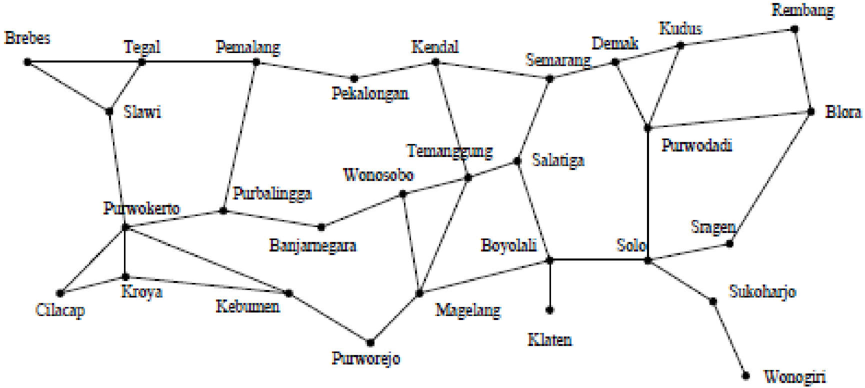

Motivation and Informal Terminologies

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3. How many different routes that can be used by a person from Cilacap to Rembang if the number of cities that is passed through must be as minimum as possible?

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By graph modeling, cities are viewed as dot or node or vertex (plural: *vertices*) while roads are viewed as edge or line or arc.

A graph usually consists of two sets, namely a set of vertices (denoted as V) and a set of edges (denoted as E).

Definition (Informal Definition of Graph)

A graph is a math structure that consists of a set of vertices and a set of edges connecting the vertices.

Contents

Directed Graph with Multiple Edges

Definition (a directed graph with multiple edges)

A graph G is denoted as a triple (V, E, f) where

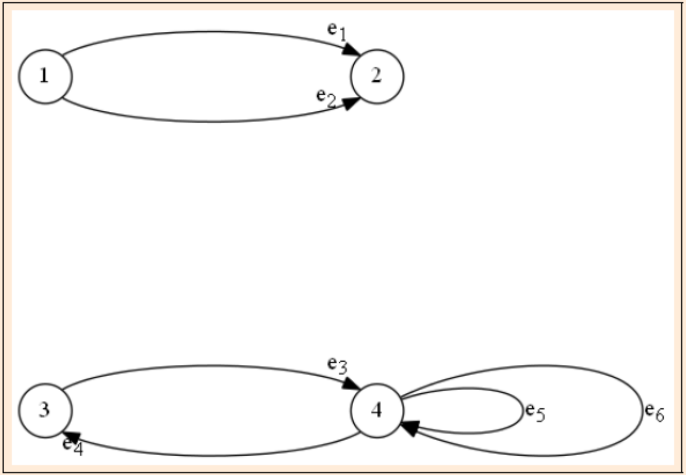
1. V is a set of all vertices in the graph,
2. E is a set of all edges in the graph,
3. f is a total function from E to $V \times V$.

A **directed** graph that has multiple edges as well as loop is called arbitrary directed graph or directed multigraph.

Exercise 1: Directed Graph with Multiple Edges

Exercise

Write the following graph in a triple (V, E, f).



Solution of Exercise 1

We have $G = (V, E, f)$ where

1. $V =$

Solution of Exercise 1

We have $G = (V, E, f)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E =$

Solution of Exercise 1

We have $G = (V, E, f)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$,
3. $f : E \to V \times V$ with the definition:

Solution of Exercise 1

We have $G = (V, E, f)$ where

1. $V = \{1, 2, 3, 4\}$,
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   - $f(e_1) = f(e_2) = (1, 2)$
   - $f(e_3) = (3, 4)$
   - $f(e_4) = (4, 3)$
   - $f(e_5) = f(e_6) = (4, 4)$.

Undirected Graph with Multiple Edges

Definition (an undirected graph with multiple edges)

A graph G is denoted as a triple (V, E, f) where

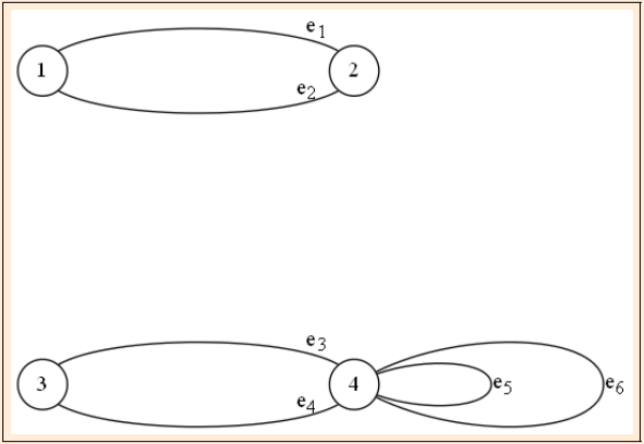
1. V is a set of all vertices in the graph,
2. E is a set of all edges in the graph,
3. f is a total function from E to a set $\{\{u, v\} \mid u, v \in V\}$.

An **undirected** graph that has multiple edges as well as loop is called a pseudograph.

Exercise 2: Undirected Graph with Multiple Edges

Exercise

Write the following graph in a triple (V, E, f)



Solution of Exercise 2

We have $G = (V, E, f)$ where

1. $V =$

Solution of Exercise 2

We have $G = (V, E, f)$ where

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1. $V = \{1, 2, 3, 4\}$,
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Solution of Exercise 2

We have $G = (V, E, f)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$,
3. $f : E \to \{\{u, v\} : u, v \in V\}$ with definition:
   - $f(e_1) = f(e_2) = \{1, 2\} = \{2, 1\}$
   - $f(e_3) = f(e_4) = \{3, 4\} = \{4, 3\}$
   - $f(e_5) = f(e_6) = \{4, 4\} = \{4\}$.

Directed Graph Without Multiple Edges

Definition (multiple edges and loop)

Based on graph definition as mentioned before, edges $e_1, e_2 \in E$ are called as parallel edge if $f(e_1) = f(e_2)$. Edge $e \in E$ is called as loop if $f(e) = (u, u)$ or $f(e) = \{u, u\} = \{u\}$.

Definition (directed graph **without** multiple edges)

A graph G is denoted as a pair (V, E) where

1. V is a set of vertices in the graph,
2. $E \subseteq V \times V$.

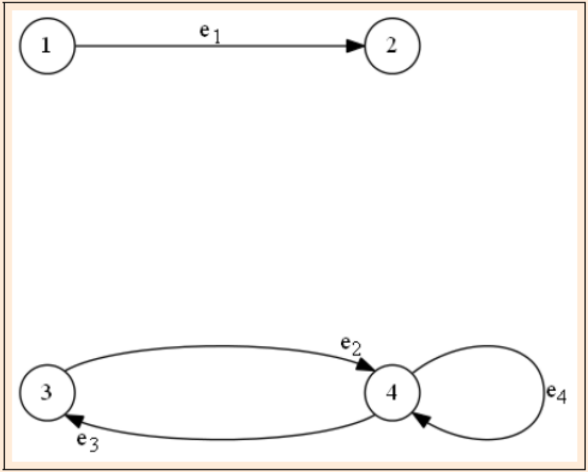
A **directed** graph that has no multiple edges but may have loop is called a digraph (*directed graph*/ *digraph*).

We have already discussed digraphs when we discussed about relation before the midterm.

Exercise 3: Directed Graph Without Multiple Edges

Exercise

Write the following graph in a pair (V, E)



Solution of Exercise 3

We have $G = (V, E)$ where

1. $V =$

Solution of Exercise 3

We have $G = (V, E)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E =$

Solution of Exercise 3

We have $G = (V, E)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E = \{(1, 2), (3, 4), (4, 3), (4, 4)\}$.

Undirected Graph Without Multiple Edges

Definition (undirected graph without multiple edges)

A graph G is denoted as a pair (V, E) where

1. V is a set of all vertices in the graph,
2. $E \subseteq \{\{u, v\} \mid u, v \in V\}$.

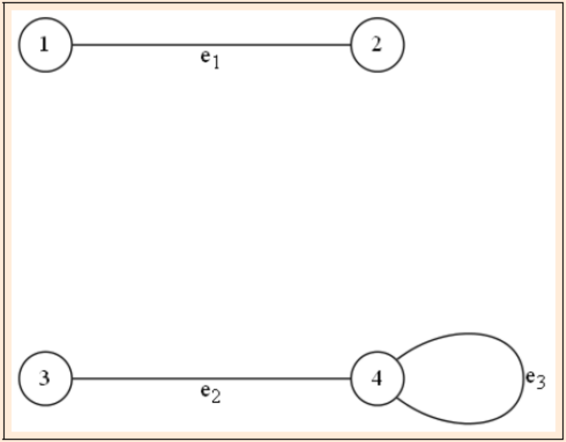
Definition (simple graph)

A simple graph is an undirected graph that has neither multiple edges nor loops.

Exercise 4: Undirected Graph without multiple edges

Exercise

Write the following graph in a pair (V, E).



Solution of Exercise 4

We have $G = (V, E)$ where

1. $V =$

Solution of Exercise 4

We have $G = (V, E)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E =$

Solution of Exercise 4

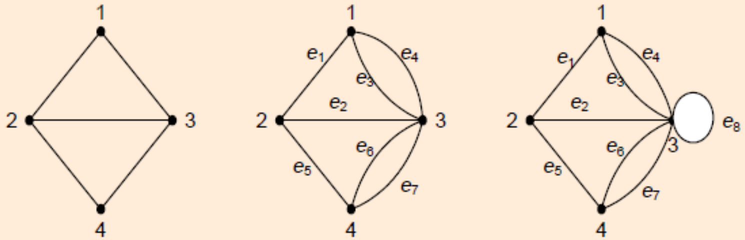
We have $G = (V, E)$ where

1. $V = \{1, 2, 3, 4\}$,
2. $E = \{\{1, 2\}, \{3, 4\}, \{4\}\}$.

Exercise 5

Exercise

Suppose G_1, G_2, and G_3 are the following graphs (from left to right respectively: G_1, G_2, and G_3).



Give formal definitions for the above graphs.

Solution of Exercise 5

1. $G_1 = (V_1, E_1)$ where $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$.

Solution of Exercise 5

1. $G_1 = (V_1, E_1)$ where $V_1 = \{1, 2, 3, 4\}$ and $E_1 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$.
2. $G_2 = (V_2, E_2, f_2)$ where $V_2 = \{1, 2, 3, 4\}$, $E_2 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and f_2 are defined as:
    1. $f_2(e_1) = \{1,2\} = \{2,1\}$
    2. $f_2(e_2) = \{2,3\} = \{3,2\}$
    3. $f_2(e_3) = f_2(e_4) = \{1,3\} = \{3,1\}$
    4. $f_2(e_5) = \{2,4\} = \{4,2\}$
    5. $f_2(e_6) = f_2(e_7) = \{3,4\} = \{4,3\}$.

Solution of Exercise 5

1. $G_1 = (V_1, E_1)$ where $V_1 = \{1,2,3,4\}$ and $E_1 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$.
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    4. $f_2(e_5) = \{2,4\} = \{4,2\}$
    5. $f_2(e_6) = f_2(e_7) = \{3,4\} = \{4,3\}$.
3. $G_3 = (V_3, E_3, f_3)$ where $V_3 = \{1,2,3,4\}$, $E_3 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, and f_3 are defined as:
    1. $f_3(e_1) = \{1,2\} = \{2,1\}$
    2. $f_3(e_2) = \{2,3\} = \{3,2\}$
    3. $f_3(e_3) = f_3(e_4) = \{1,3\} = \{3,1\}$
    4. $f_3(e_5) = \{2,4\} = \{4,2\}$
    5. $f_3(e_6) = f_3(e_7) = \{3,4\} = \{4,3\}$
    6. $f_3(e_8) = \{3,3\} = \{3\}$.

Finite and Infinite graph

We already know that we can write a graph in a formal definition $G = (V, E, f)$ or $G = (V, E)$, set V is a set of vertices and set E is a set of edges.

Definition (Finite Graph and Infinite Graph)

A graph $G = (V, E, f)$ or $G = (V, E)$ is called a finite graph if V is a finite set, in other words $|V| = n$ for an $n \in \mathbb{N}$. If V is infinite, then G is called an infinite graph.

Notes

In this course, every graph is assumed to be a finite graph.

Contents

Adjacency, Neighbor, and Neighborhood

Definition (adjacent and incident in undirected graphs)

1. Suppose $G = (V, E, f)$, $v_1, v_2 \in V$ is called **adjacent** if there is $e \in E$ with properties $f(e) = \{v_1, v_2\}$.
2. Suppose $G = (V, E)$, $v_1, v_2 \in V$ is called **adjacent** if $\{v_1, v_2\} \in E$.

If $f(e) = \{v_1, v_2\}$ (or $e = \{v_1, v_2\}$) then e is called **incident** with v_1 and v_2. Then vertices v_1 and v_2 are called as **endpoints** of edge $e \in E$.

Definition (neighbourhood in undirected graphs)

Suppose $G = (V, E, f)$, $u \in V$ is called as neighbor of $v \in V$ if there is $e \in E$ such that $f(e) = \{u, v\}$. Neighborhood of v, is denoted by $N(v)$, defined as a set of all adjacent vertices of v.

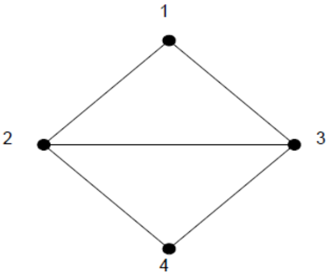
Definition (adjacency in directed graphs)

1. Suppose $G = (V, E, f)$ is a directed graph. A vertex v_1 is called adjacent to v_2 or vertex v_2 is called adjacent from v_1 if $f(e) = (v_1, v_2)$ for a $e \in E$.
2. Suppose $G = (V, E)$ is a directed graph. A vertex v_1 is called adjacent to v_2 or a vertex v_2 is called adjacent from v_1 if $(v_1, v_2) \in E$.

If $f(e) = (v_1, v_2)$ (or $e = (v_1, v_2)$) then v_1 is called an initial vertex and v_2 is called a terminal vertex of edge $e \in E$.

Adjacency Illustration

Suppose G is a simple undirected graph as follows.



We have:

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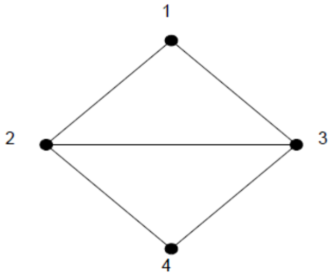

We have:

1. vertices 1 and 2 are adjacent to one another, as well as vertices 1 and 3, 2 and 3, 2 and 4, also 3 and 4;

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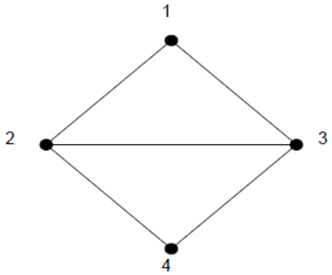


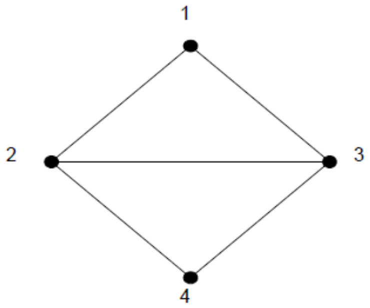
We have:

1. vertices 1 and 2 are adjacent to one another, as well as vertices 1 and 3, 2 and 3, 2 and 4, also 3 and 4;
2. vertices 1 and 4 are non-adjacent, because there is no edge connecting vertices 1 and 4.

In an undirected graph, vertices a and b are adjacent if there is an edge connecting them.

Neighborhood Illustration

Suppose G is a simple undirected graph as follows.



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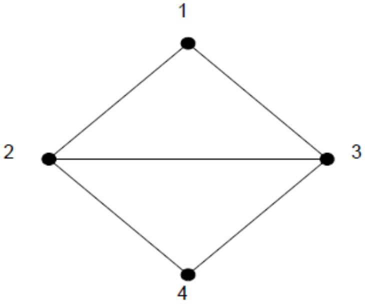

We have:

1. $N(1) = \{2, 3\}$, where $\{2, 3\}$ is the neighborhood of vertex 1 because there is an edge connecting vertex 1 and vertex 2 and also an edge connecting vertex 1 and vertex 3;

Neighborhood Illustration

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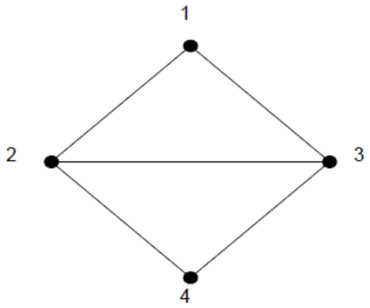
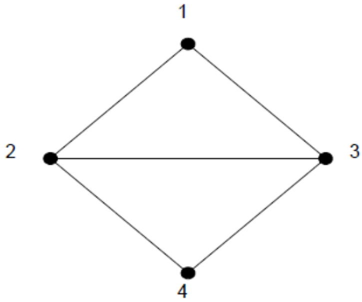


We have:

1. $N(1) = \{2, 3\}$, where $\{2, 3\}$ is the neighborhood of vertex 1 because there is an edge connecting vertex 1 and vertex 2 and also an edge connecting vertex 1 and vertex 3;
2. $N(2) = \{1, 3, 4\}$, where $\{1, 3, 4\}$ is the neighborhood of vertex 2 because there is an edge connecting vertex 2 and vertex 1, vertex 2 and vertex 3, and vertex 2 with vertex 4.

Incident Illustration

Suppose G is a simple undirected graph as follows.



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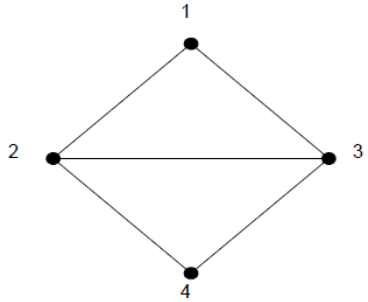

We have:

1. edge $\{1, 2\}$ is incident on vertex 1 as well as on vertex 2, edge $\{1, 3\}$ is incident on vertex 1 as well as vertex 3;

Incident Illustration

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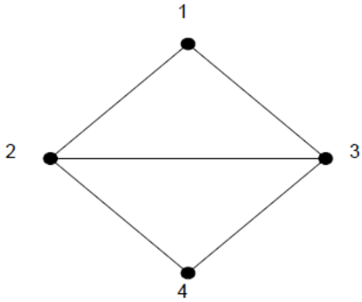

We have:

1. edge $\{1, 2\}$ is incident on vertex 1 as well as on vertex 2, edge $\{1, 3\}$ is incident on vertex 1 as well as vertex 3;
2. edge $\{1, 2\}$ is not incident on vertex 3 as well as vertex 4.

In a simple undirected graph, edge $\{a, b\}$ is incident on vertex a as well as vertex b.

Degree of a Vertex in Undirected Graphs

Definition (degree of a vertex in undirected graphs)

Suppose $G = (V, E, f)$ is an undirected graph. Degree of a vertex $v \in V$ in G is the number of edges that incident with vertex v, except that a loop at a vertex contributes twice to the degree of that vertex. Degree of v is denoted as $\deg(v)$.

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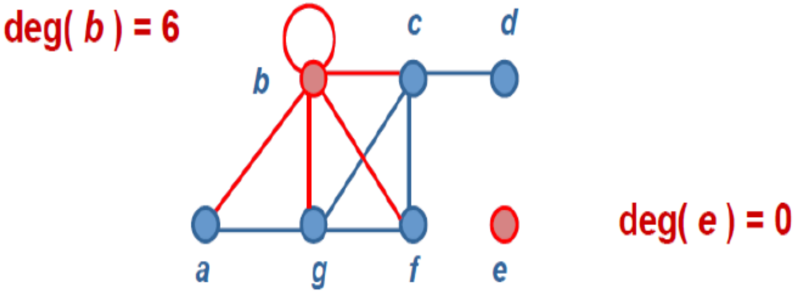
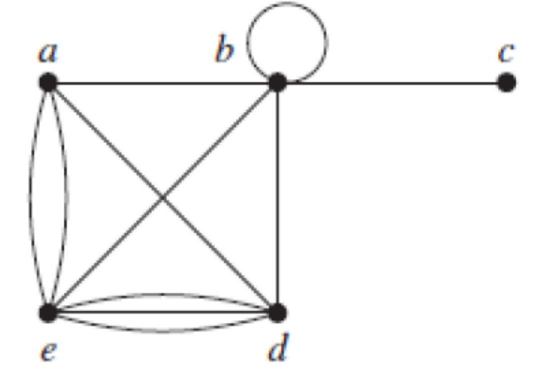


Exercise 6: Determining Neighborhood and Degree of a Vertex

Exercise

Determine the neighborhood and degree of each vertex in following graph G



Isolated Vertex and Pendant

Definition (isolated vertex and pendant)

1. If $G = (V, E, f)$ is an undirected graph, then vertex $v \in V$ is called as isolated vertex if $\deg(v) = 0$.
2. If $G = (V, E, f)$ is an undirected graph, then vertex $v \in V$ is called as a pendant if $\deg(v) = 1$.

Isolated Vertex and Pendant

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1. If $G = (V, E, f)$ is an undirected graph, then vertex $v \in V$ is called as isolated vertex if $\deg(v) = 0$.
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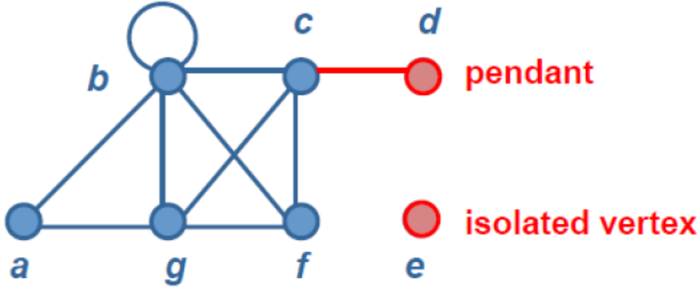
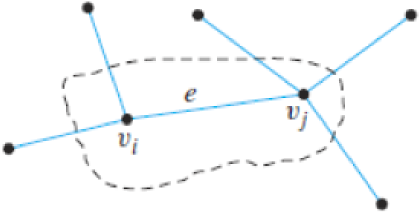


Handshaking Theorem (for Undirected Graph)

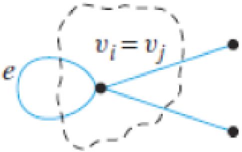
Theorem (Handshaking Theorem)

If $G = (V, E, f)$ is an undirected graph, then $2|E| = \sum_{v \in V} \deg(v)$.

Illustration of Handshaking Theorem proof's.



$i \neq j$

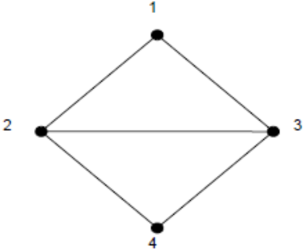


$i = j$

Corollary

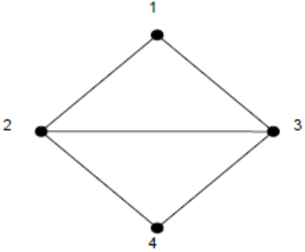
Every undirected graph $G = (V, E, f)$ has an even number of vertex with odd degree.

Illustration of Handshaking Theorem in Undirected Graph



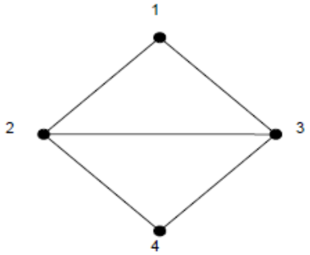
Suppose the graph above is graph G_1. We have: $\deg(1) =$

Illustration of Handshaking Theorem in Undirected Graph



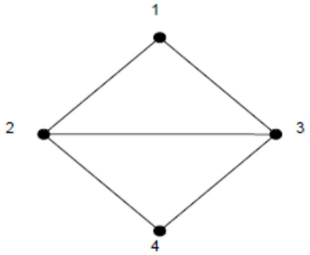
Suppose the graph above is graph G_1. We have: $\deg(1) = 2$,
$\deg(2) = \deg(3) =$

Illustration of Handshaking Theorem in Undirected Graph



Suppose the graph above is graph G_1. We have: $\deg(1) = 2$, $\deg(2) = \deg(3) = 3$, and $\deg(4) =$

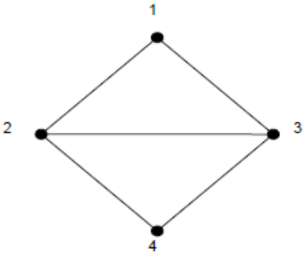
Illustration of Handshaking Theorem in Undirected Graph



Suppose the graph above is graph G_1. We have: $\deg(1) = 2$, $\deg(2) = \deg(3) = 3$, and $\deg(4) = 2$. The number of edges is 5. We have

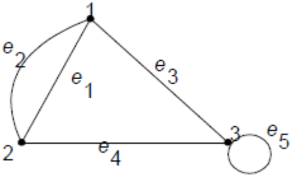
$$\begin{aligned} |E| &= 5 \\ \sum_{v \in V} \deg(v) &= \end{aligned}$$

Illustration of Handshaking Theorem in Undirected Graph

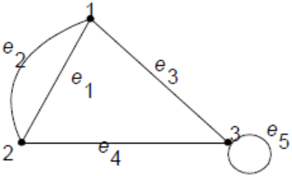


Suppose the graph above is graph G_1. We have: $\deg(1) = 2$, $\deg(2) = \deg(3) = 3$, and $\deg(4) = 2$. The number of edges is 5. We have

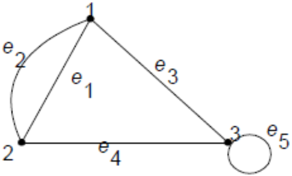
$$
\begin{aligned}
|E| &= 5 \\
\sum_{v \in V} \deg(v) &= \deg(1) + \deg(2) + \deg(3) + \deg(4) \\
&= 2 + 3 + 3 + 2 = 10, \text{ that is} \\
2|E| &= \sum_{v \in V} \deg(v).
\end{aligned}
$$



Suppose the graph above is graph G_2. We have: $\deg(1) =$



Suppose the graph above is graph G_2. We have: $\deg(1) = 3$, $\deg(2) =$



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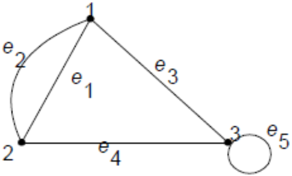

Suppose the graph above is graph G_2. We have: $\deg(1) = 3$, $\deg(2) = 3$, $\deg(3) = 4$. The number of edges is 5. We have

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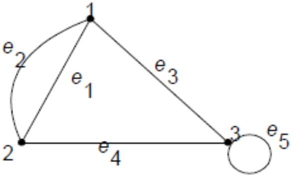
\begin{aligned}
|E| &= 5 \\
\sum_{v \in V} \deg(v) &=
\end{aligned}
$$



Suppose the graph above is graph G_2. We have: $\deg(1) = 3$, $\deg(2) = 3$, $\deg(3) = 4$. The number of edges is 5. We have

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\begin{aligned}
|E| &= 5 \\
\sum_{v \in V} \deg(v) &= \deg(1) + \deg(2) + \deg(3) \\
&= 3 + 3 + 4 = 10, \text{ that is} \\
2|E| &= \sum_{v \in V} \deg(v).
\end{aligned}
$$

Exercise 7: Implementation of Handshaking Theorem

Exercise

Check whether we can draw the following graphs.

1. Graph $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d, e\}$ and $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 1$, $\deg(d) = 1$, and $\deg(e) = 2$.
2. Graph $G_2 = (V_2, E_2)$ where $V_2 = \{a, b, c, d, e\}$ and $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 3$, $\deg(d) = 4$, and $\deg(e) = 4$.

Solution:

Exercise 7: Implementation of Handshaking Theorem

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Check whether we can draw the following graphs.

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Solution:

1. Notice that $\sum_{v \in V_1} \deg(v) = 2 + 3 + 1 + 1 + 2 = 9$.

Exercise 7: Implementation of Handshaking Theorem

Exercise

Check whether we can draw the following graphs.

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Solution:

1. Notice that $\sum_{v \in V_1} \deg(v) = 2 + 3 + 1 + 1 + 2 = 9$. By handshaking theorem $2|E_1| = 9$, hence $|E_1| = \frac{9}{2} \notin \mathbb{N}_0$.

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Check whether we can draw the following graphs.

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Solution:

1. Notice that $\sum_{v \in V_1} \deg(v) = 2 + 3 + 1 + 1 + 2 = 9$. By handshaking theorem $2|E_1| = 9$, hence $|E_1| = \frac{9}{2} \notin \mathbb{N}_0$. Therefore, there is no graph G_1 that satisfies the criteria.
2. Notice that $\sum_{v \in V_2} \deg(v) = 2 + 3 + 3 + 4 + 4 = 16$.

Exercise 7: Implementation of Handshaking Theorem

Exercise

Check whether we can draw the following graphs.

1. Graph $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d, e\}$ and $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 1$, $\deg(d) = 1$, and $\deg(e) = 2$.
2. Graph $G_2 = (V_2, E_2)$ where $V_2 = \{a, b, c, d, e\}$ and $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 3$, $\deg(d) = 4$, and $\deg(e) = 4$.

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2. Notice that $\sum_{v \in V_2} \deg(v) = 2 + 3 + 3 + 4 + 4 = 16$. By handshaking theorem $2|E_2| = 16$, hence $|E_2| = 8$.

Exercise 7: Implementation of Handshaking Theorem

Exercise

Check whether we can draw the following graphs.

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2. Graph $G_2 = (V_2, E_2)$ where $V_2 = \{a, b, c, d, e\}$ and $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 3$, $\deg(d) = 4$, and $\deg(e) = 4$.

Solution:

1. Notice that $\sum_{v \in V_1} \deg(v) = 2 + 3 + 1 + 1 + 2 = 9$. By handshaking theorem $2|E_1| = 9$, hence $|E_1| = \frac{9}{2} \notin \mathbb{N}_0$. Therefore, there is no graph G_1 that satisfies the criteria.
2. Notice that $\sum_{v \in V_2} \deg(v) = 2 + 3 + 3 + 4 + 4 = 16$. By handshaking theorem $2|E_2| = 16$, hence $|E_2| = 8$. Therefore, G_2 is a graph with 8 edges.

Degree of a Vertex in Directed Graph

Definition (degree of a vertex in directed graph)

Suppose $G = (V, E, f)$ is a multiple directed graph. If $v \in V$, then in-degree of v, is denoted as $\deg^{-}(v)$ or $\deg_{in}(v)$, is the number of edges with terminal vertex v. Out-degree of v, is denoted as $\deg^{+}(v)$ or $\deg_{out}(v)$, is the number of edges with initial vertex v.

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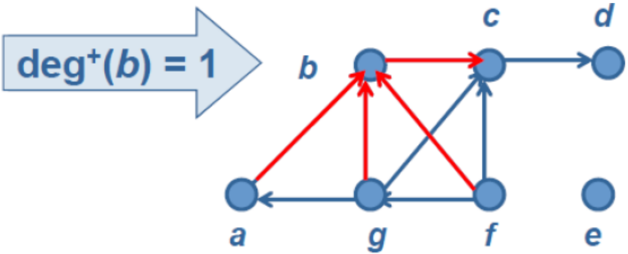
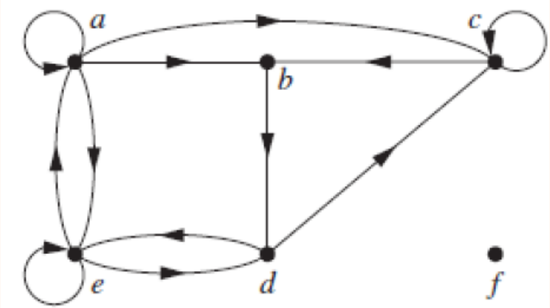


Exercise 8: Determine Degree of a Vertex Directed Graph

Exercise

Determine in-degree and out-degree for every vertex in the following graph G



Handshaking Theorem (for Directed Graph)

Theorem (Directed Handshaking Theorem)

Suppose $G = (V, E, f)$ is a multiple directed graph (or directed graph), then

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|, \text{ or}$$
$$\sum_{v \in V} \deg_{in}(v) = \sum_{v \in V} \deg_{out}(v) = |E|.$$

Contents

Subgraph and Spanning Subgraph

Definition (subgraph and spanning subgraph)

Suppose $G = (V, E)$ is an undirected graph without multiple edges.

1. Graph $H = (W, F)$ is called as a subgraph of G if $W \subseteq V$ and $F \subseteq E$.
2. Graph H is called a proper subgraph of G if H is a subgraph of G and $H \neq G$.
3. Furthermore, a subgraph $H = (W, F)$ of graph $G = (V, E)$ is called as spanning subgraph of G if $W = V$.

Suppose G is a graph as follows

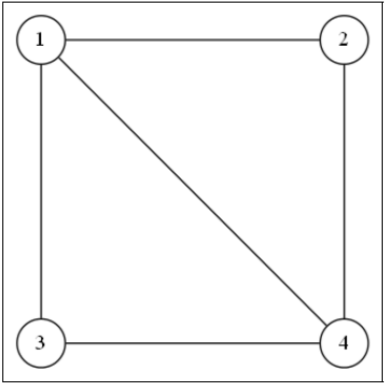


Figure: Graph G

Suppose H_1 is a graph as follows.

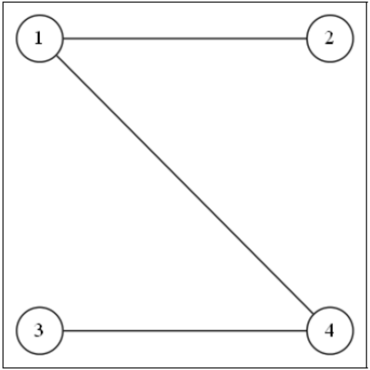


Figure: Graph H_1

Is H_1 a subgraph of G? Is H_1 a spanning subgraph of G?

Suppose H_1 is a graph as follows.

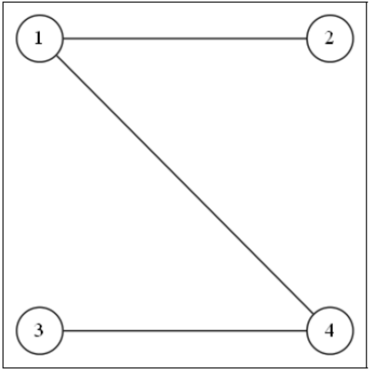


Figure: Graph H_1

Is H_1 a subgraph of G? Is H_1 a spanning subgraph of G? Graph H_1 is a subgraph and spanning subgraph of G.

Suppose H_2 is the following graph.

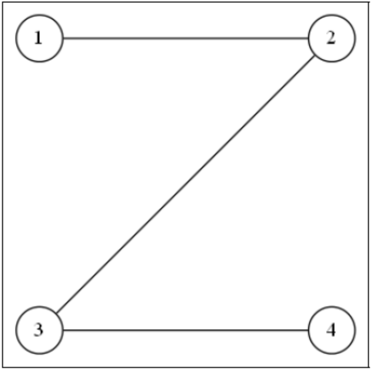


Figure: Graph H_2

Is H_2 a subgraph of G? Is H_2 a spanning subgraph of G?

Suppose H_2 is the following graph.

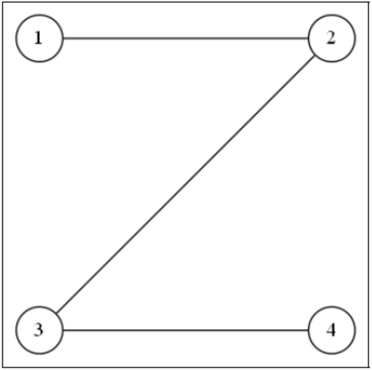


Figure: Graph H_2

Is H_2 a subgraph of G? Is H_2 a spanning subgraph of G? Graph H_2 is not a subgraph and not a spanning subgraph of G because edge $\{2, 3\}$ is not an edge on G.

Suppose H_3 is the following graph.

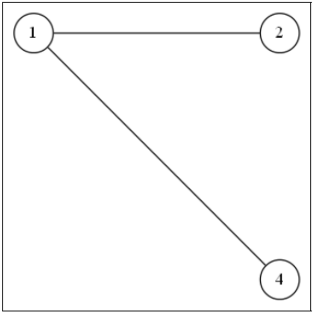


Figure: Graph H_3

Is H_3 a subgraph of G? Is H_3 a spanning subgraph of G?

Suppose H_3 is the following graph.

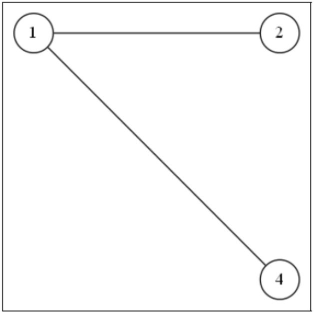


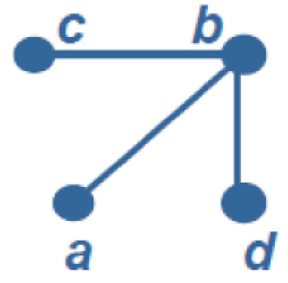
Figure: Graph H_3

Is H_3 a subgraph of G? Is H_3 a spanning subgraph of G? Graph H_3 is a subgraph of G but not a spanning subgraph of G (because the set of vertices for H_3 and G is different).

Exercise 9: Determine the number of spanning subgraph

Exercise

Determine the number of different spanning subgraphs of the following graph



Hint: You do not need to draw all spanning subgraphs of the graph.

Complement Graph

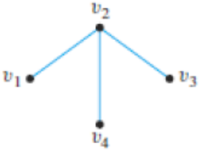
Definition (Complement Graph)

Suppose $G = (V_G, E_G)$ is a graph. Graph $\bar{G} = (V_{\bar{G}}, E_{\bar{G}})$ is a complement of graph G if

1. $V_{\bar{G}} = V_G$
2. u and v are two vertices adjacent in G if and only if u and v are not adjacent $\bar{G}$, formally

$$\{u, v\} \in E_G \Leftrightarrow \{u, v\} \notin E_{\bar{G}} \quad \text{(for undirected graph)}$$
$$(u, v) \in E_G \Leftrightarrow (u, v) \notin E_{\bar{G}} \quad \text{(for directed graph).}$$

Suppose G is the following graph.



Then $\bar{G}$ is the following graph.

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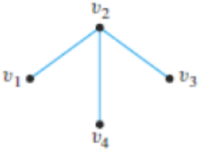

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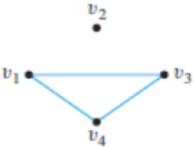

Subgraph Complement

Definition (Subgraph Complement)

Suppose $G = (V, E)$ is a graph and $G_1 = (V_1, E_1)$ is a subgraph of G. Complement of subgraph G_1 of graph G is graph $G_2 = (V_2, E_2)$ with the properties:

1. $E_2 = E \setminus E_1$;
2. $V_2 \subseteq V$ is a set of vertices with the properties: elements of E_2 are incident on the vertices in V_2.

The following is an illustration of subgraph complement.

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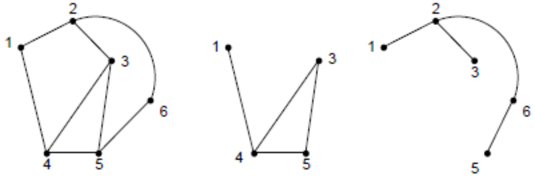

The graph in the middle $G_1 = (V_1, E_1)$ is the subgraph of the leftmost graph $G = (V, E)$.

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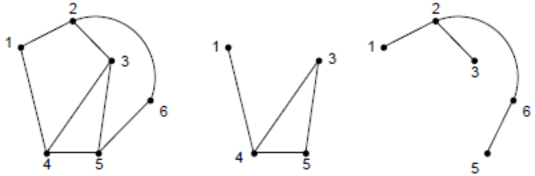


The graph in the middle $G_1 = (V_1, E_1)$ is the subgraph of the leftmost graph $G = (V, E)$. The rightmost graph is $G_2 = (V_2, E_2)$ and is a subgraph complement of G_1 to the graph G.

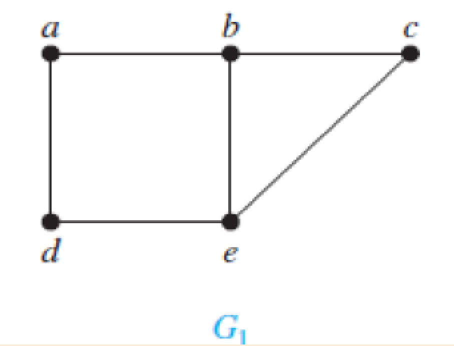
Graph Union from Two Simple Graphs

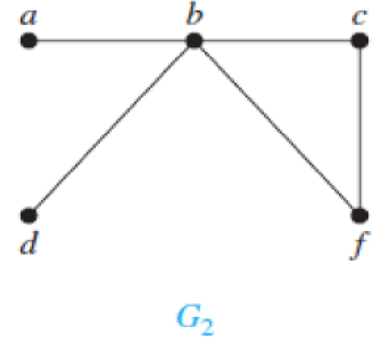
Definition

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two simple graphs (undirected, has no multiple edges, has no loop). Graph union of G_1 and G_2, is denoted as $G_1 \cup G_2$, is a graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Exercise

Draw $G_1 \cup G_2$ if G_1 and G_2 are the following graphs





Contents

Complete Graph K_n

Remember: a simple graph is an undirected graph that **has no** multiple edges and **has no** loop.

Definition

Suppose n is an integer, $n = 1, 2, \ldots$. A complete graph with n vertices, denoted as K_n, is a graph where each of vertex is adjacent to one another.

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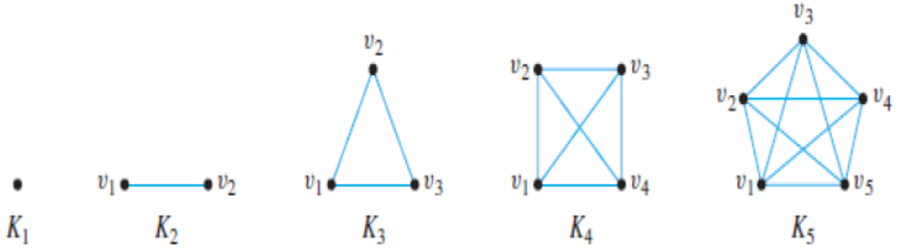

Circle/Cyclic Graph C_n

Definition

A circle or a cyclic graph with n vertices $(n \geq 3)$, denoted as C_n, is a graph with its set of vertices $\{v_1, v_2, \ldots, v_n\}$ and its set of edges

$$\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}.$$

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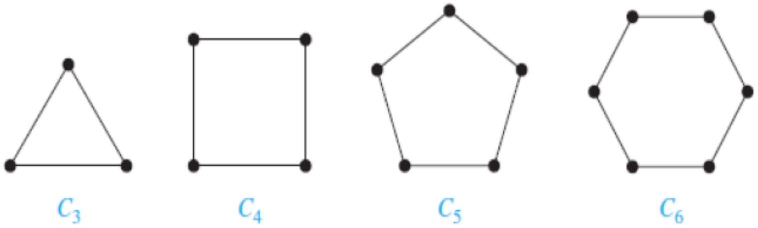

Wheel Graph W_n

Definition

A wheel graph with $n + 1$ vertices $(n \geq 3)$, denoted as W_n, is a graph that is obtained by adding one vertex v_{n+1} on graph C_n such that v_{n+1} is adjacent with every vertex in the set $\{v_1, v_2, \ldots, v_n\}$.

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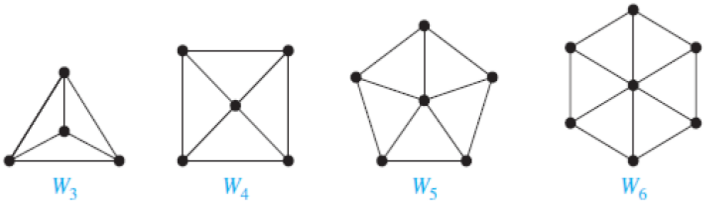

Regular Graph

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A simple graph is called a regular graph if every vertex on the graph has identical degree.

This is the example of regular graph with 4 vertices and each vertex has degree 3.

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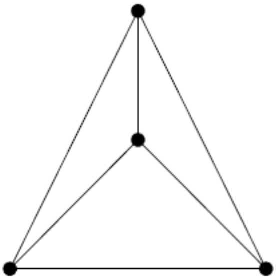


Bipartite Graph

Definition

A bipartite graph $G = (V, E)$ is a graph that satisfies the following properties

1. $V = V_1 \cup V_2$ where
   1. $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$,
   2. $V_1 \cap V_2 = \emptyset$.

   In other words V_1 and V_2 are **partition** on the set V.
2. $\{u_1, u_2\} \in E$ if and only if exactly one of the two following conditions is satisfied
   1. $u_1 \in V_1$ and $u_2 \in V_2$, or
   2. $u_2 \in V_1$ and $u_1 \in V_2$.

   In other words every edge connecting two vertices on different partition .

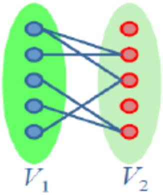
Complete Bipartite Graph

Definition

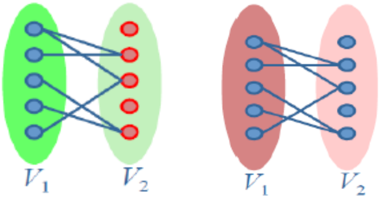
A graph is called a **complete** bipartite graph $K_{m,n}$ if $K_{m,n} = (V, E)$ where

1. V can be partitioned into V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$.
2. $E = \{\{v_1, v_2\} : v_1 \in V_1 \text{ and } v_2 \in V_2\}$, in other words every vertex on V_1 is adjacent with any vertex on V_2.

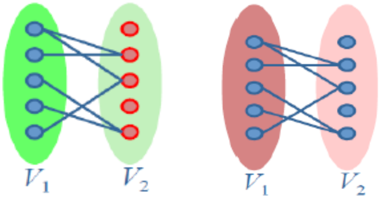
Examples of Bipartite Graph

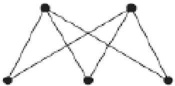


Examples of Bipartite Graph

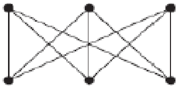


Examples of Bipartite Graph

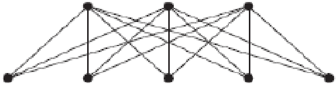




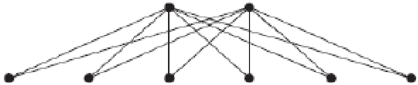
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

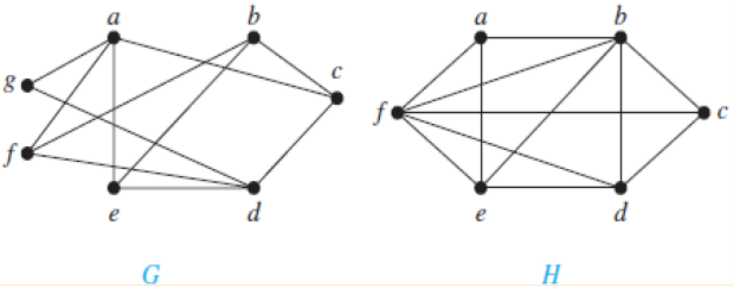


$K_{2,6}$

Exercise 10: Simple Graphs with Special Structure

Exercise

1. Determine the number of edges on K_{2019}.
2. Determine the number of edges on C_{2019}.
3. Determine the number of edges on W_{2019}.
4. Determine the number of edges on $K_{2019,2020}$.
5. Check whether the following graph is a bipartite graph.



Contents

Adjacency Matrix

Definition

1. Suppose $G = (V, E, f)$ is an undirected graph that may have multiple edges or loop with $|V| = n$. Adjacency matrix of G is a matrix $\mathbf{A}_G = [a_{ij}]$ with size $n \times n$ where the entries are as follows

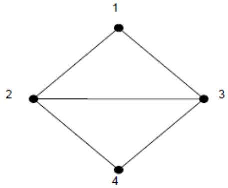
$$a_{ij} = \begin{cases} m, & \text{if } |\{e \in E \mid f(e) = \{v_i, v_j\}\}| = m. \\ 0, & \text{otherwise.} \end{cases}$$

2. Suppose $G = (V, E)$ is an undirected graph that has no multiple edges but may have loop, then

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \\ 0, & \text{otherwise.} \end{cases}$$

Definition of adjacency matrix for directed graph can use an analogy with the above definition (replace $\{v_i, v_j\}$ with (v_i, v_j)).

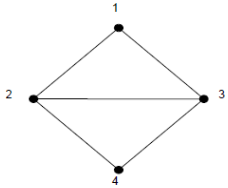
Suppose G is the following graph.



Adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G =$$

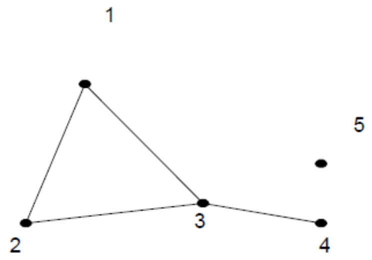
Suppose G is the following graph.



Adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

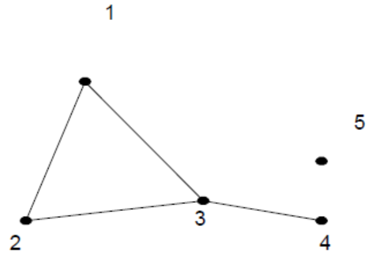
Suppose G is the following graph.



The adjacency matrix for graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G =$$

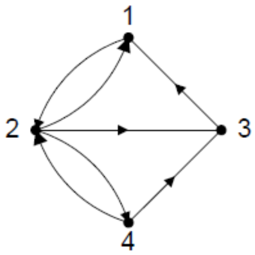
Suppose G is the following graph.



The adjacency matrix for graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

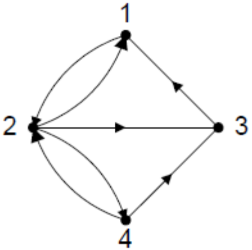
Suppose G is the following graph.



Adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G =$$

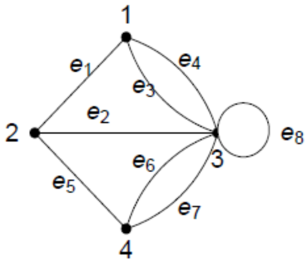
Suppose G is the following graph.



Adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

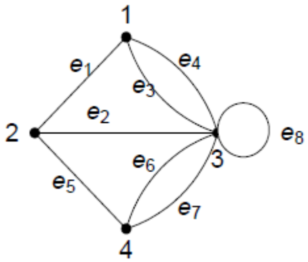
Suppose G is the following graph.



The adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G =$$

Suppose G is the following graph.



The adjacency matrix of graph G is $\mathbf{A}_G$, where

$$\mathbf{A}_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Determining Degree of a Vertex from Adjacency Matrix

Vertex's Degree from Adjacency Matrix

Suppose $\mathbf{A}_G = [a_{ij}]$ is an adjacency matrix of an undirected graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ that contains no loop, then

$$\deg(v_i) = \sum_{j=1}^{n} a_{ij}.$$

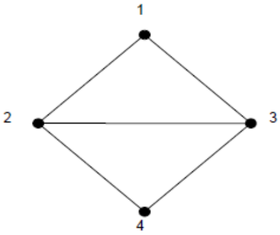
Suppose $\mathbf{A}_G = [a_{ij}]$ is an adjacency matrix of a directed graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\deg_{in}(v_i) = \deg^{-}(v_i) = \text{sum of the values on column } i = \sum_{j=1}^{n} a_{ji}$$

$$\deg_{out}(v_i) = \deg^{+}(v_i) = \text{sum of the values on row } i = \sum_{j=1}^{n} a_{ij}$$

Example on Undirected Graphs

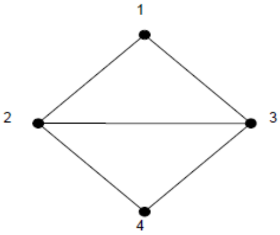
Suppose G is the following graph.



We have $\mathbf{A}_G =$

Example on Undirected Graphs

Suppose G is the following graph.

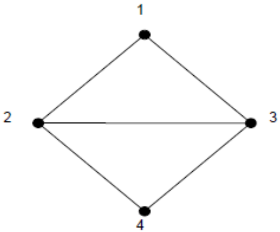


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg(2) =$

Example on Undirected Graphs

Suppose G is the following graph.

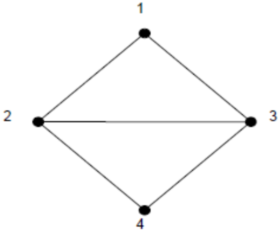


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} =$

Example on Undirected Graphs

Suppose G is the following graph.

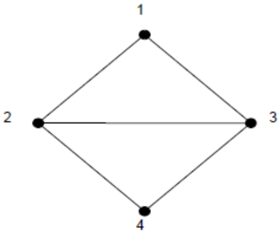


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} = 1 + 0 + 1 + 1 = 3.$
- $\deg(4) =$

Example on Undirected Graphs

Suppose G is the following graph.

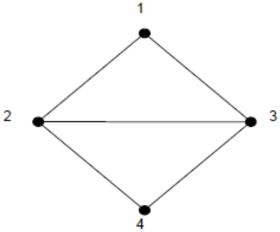


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} = 1 + 0 + 1 + 1 = 3.$
- $\deg(4) = \sum_{j=1}^{4} a_{4j} = a_{41} + a_{42} + a_{43} + a_{44} =$

Example on Undirected Graphs

Suppose G is the following graph.

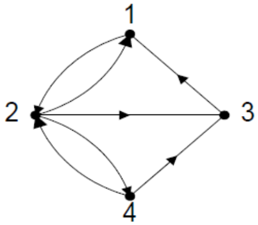


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} = 1 + 0 + 1 + 1 = 3.$
- $\deg(4) = \sum_{j=1}^{4} a_{4j} = a_{41} + a_{42} + a_{43} + a_{44} = 0 + 1 + 1 + 0 = 2.$

Example on a Directed Graph

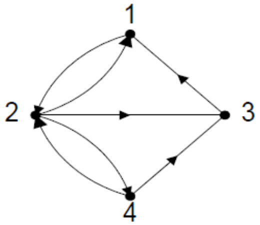
Suppose G is the following graph.



We have $\mathbf{A}_G =$

Example on a Directed Graph

Suppose G is the following graph.

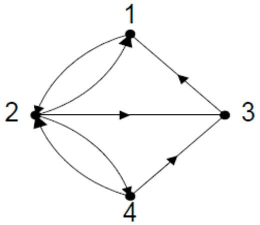


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg_{in}(2) = \deg^{-}(2) =$

Example on a Directed Graph

Suppose G is the following graph.

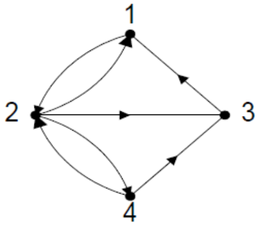


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg_{in}(2) = \deg^{-}(2) = \sum_{j=1}^{4} a_{j2} = a_{12} + a_{22} + a_{32} + a_{42} =$

Example on a Directed Graph

Suppose G is the following graph.

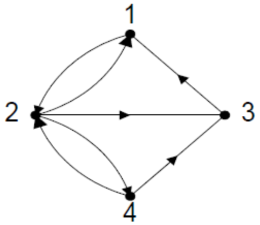


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg_{in}(2) = \deg^{-}(2) = \sum_{j=1}^{4} a_{j2} = a_{12} + a_{22} + a_{32} + a_{42} = 1 + 0 + 0 + 1 = 2.$
- $\deg_{out}(2) = \deg^{+}(2) =$

Example on a Directed Graph

Suppose G is the following graph.

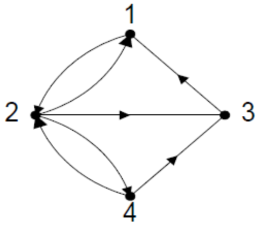


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg_{in}(2) = \deg^{-}(2) = \sum_{j=1}^{4} a_{j2} = a_{12} + a_{22} + a_{32} + a_{42} = 1 + 0 + 0 + 1 = 2.$
- $\deg_{out}(2) = \deg^{+}(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} =$

Example on a Directed Graph

Suppose G is the following graph.

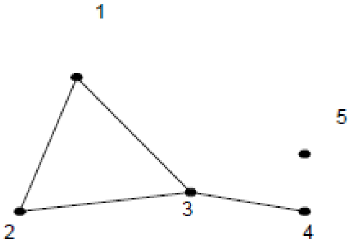


We have $\mathbf{A}_G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Therefore

- $\deg_{in}(2) = \deg^{-}(2) = \sum_{j=1}^{4} a_{j2} = a_{12} + a_{22} + a_{32} + a_{42} = 1 + 0 + 0 + 1 = 2.$
- $\deg_{out}(2) = \deg^{+}(2) = \sum_{j=1}^{4} a_{2j} = a_{21} + a_{22} + a_{23} + a_{24} = 1 + 0 + 1 + 1 = 3.$

Adjacency List

The adjacency list of an undirected graph is a list that explains the adjacency between a vertex **with** other vertices in its neighborhood.
Suppose G is the following graph.

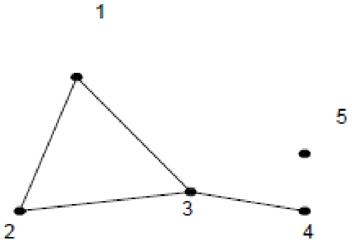


The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	

Adjacency List

The adjacency list of an undirected graph is a list that explains the adjacency between a vertex **with** other vertices in its neighborhood.
Suppose G is the following graph.

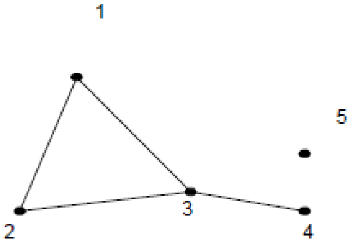


The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	$2, 3$
2	

Adjacency List

The adjacency list of an undirected graph is a list that explains the adjacency between a vertex **with** other vertices in its neighborhood.
Suppose G is the following graph.



The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	$2, 3$
2	$1, 3$
3	

Adjacency List

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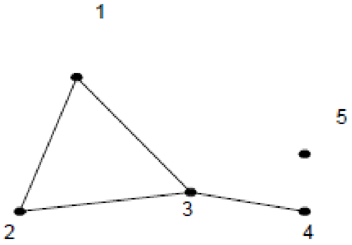
Suppose G is the following graph.

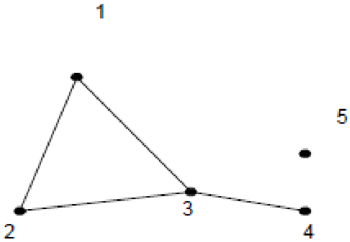


The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	$2, 3$
2	$1, 3$
3	$1, 2, 4$
4	

Adjacency List

The adjacency list of an undirected graph is a list that explains the adjacency between a vertex **with** other vertices in its neighborhood.
Suppose G is the following graph.

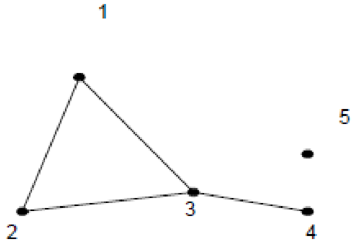


The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	$2, 3$
2	$1, 3$
3	$1, 2, 4$
4	3
5	

Adjacency List

The adjacency list of an undirected graph is a list that explains the adjacency between a vertex **with** other vertices in its neighborhood.
Suppose G is the following graph.



The adjacency list of G is as follows.

Vertex	Neighbor Vertices
1	$2, 3$
2	$1, 3$
3	$1, 2, 4$
4	3
5	- (none)

Comparison of Adjacency Matrix and Adjacency List

Adjacency matrix has some advantages:

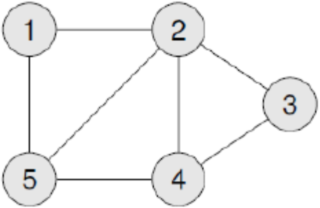
1. it is suitable for dense graph, namely a graph $G = (V, E)$ with the value of $|E|$ is approximate to $|V|^2$,
2. it can give information about the availability of an edge connecting two vertices quickly.

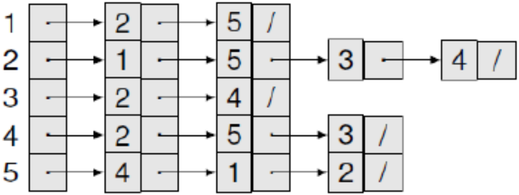
However, the use of adjacency matrix needs more storage to store a matrix that contains $|V|^2$ components.
Adjacency list has some advantages:

1. it is suitable for sparse graph, namely a graph $G = (V, E)$ with the value of $|E|$ is far less than $|V|^2$,
2. it needs less storage than adjacency matrix that contains $|V|^2$ components.

However, an adjacency list cannot give a quick information about the availability of an edge connecting two vertices.

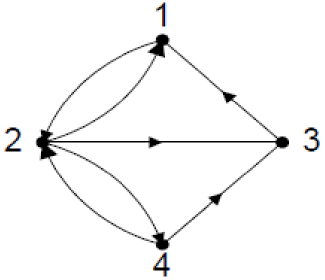
In its implementation on programming language, an adjacency list is created using *pointer* (as you learned in Data Structure Course).





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

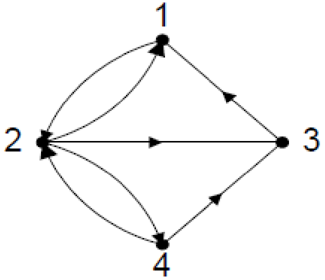
An adjacency list of a directed graph is a list that explain the adjacency between one vertex **to** another vertices in its neighborhood.
Suppose G is the following graph.



Adjacency list of G is as follows.

Initial vertex	Terminal vertex
1	

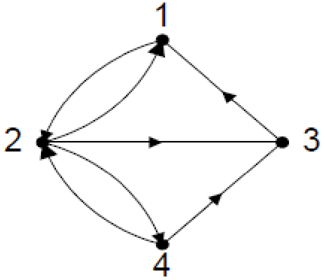
An adjacency list of a directed graph is a list that explain the adjacency between one vertex **to** another vertices in its neighborhood.
Suppose G is the following graph.



Adjacency list of G is as follows.

Initial vertex	Terminal vertex
1	2
2	

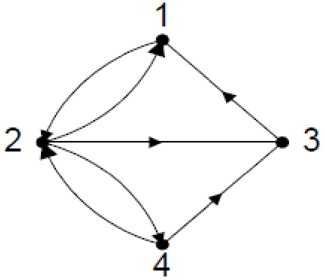
An adjacency list of a directed graph is a list that explain the adjacency between one vertex **to** another vertices in its neighborhood.
Suppose G is the following graph.



Adjacency list of G is as follows.

Initial vertex	Terminal vertex
1	2
2	$1, 3, 4$
3	

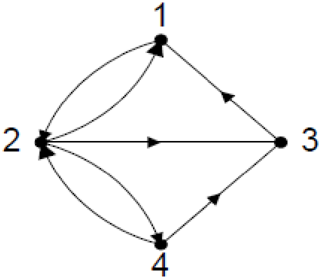
An adjacency list of a directed graph is a list that explain the adjacency between one vertex **to** another vertices in its neighborhood.
Suppose G is the following graph.



Adjacency list of G is as follows.

Initial vertex	Terminal vertex
1	2
2	$1, 3, 4$
3	1
4	

An adjacency list of a directed graph is a list that explain the adjacency between one vertex **to** another vertices in its neighborhood.
Suppose G is the following graph.



Adjacency list of G is as follows.

Initial vertex	Terminal vertex
1	2
2	$1, 3, 4$
3	1
4	$2, 3$

Incidence Matrix

Definition

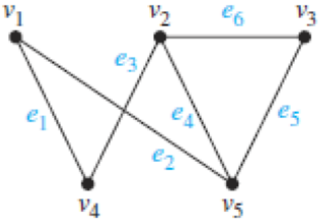
Suppose $G = (V, E, f)$ is an undirected graph that may have multiple edges or loop where $|V| = m$ and $|E| = n$. Incidence matrix of G is a matrix $\mathbf{B} = [b_{ij}]$ with size $m \times n$ where the entries are as follows

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ endpoint of } e_j \text{ and } e_j \text{ is not a loop,} \\ 2, & \text{if } v_i \text{ endpoint of } e_j \text{ and } e_j \text{ is a loop,} \\ 0, & \text{otherwise.} \end{cases}$$

If $G = (V, E, f)$ is a directed graph that may have multiple edges or loop where $|V| = m$ and $|E| = n$, the entries $\mathbf{B}$ are as follows

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ initial vertex of } e_j \text{ and } e_j \text{ is not loop,} \\ -1, & \text{if } v_i \text{ terminal vertex of } e_j \text{ and } e_j \text{ is not loop,} \\ 2, & \text{if } v_i \text{ initial vertex/ terminal vertex of } e_j \text{ and } e_j \text{ is loop,} \\ 0, & \text{otherwise.} \end{cases}$$

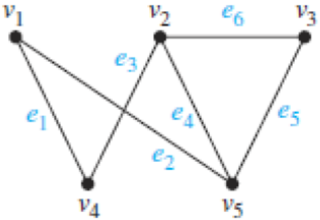
Suppose G is the following graph.



Incidence matrix for G is $\mathbf{B}_G$, where

$$\mathbf{B}_G =$$

Suppose G is the following graph.



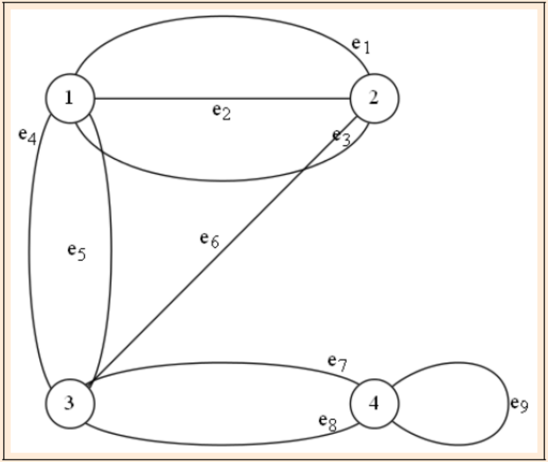
Incidence matrix for G is $\mathbf{B}_G$, where

$$\mathbf{B}_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Exercise 11: Matrix Representation for an Undirected Graph

Exercise

Determine the adjacency matrix and incidence matrix of the following graph G



Solution:

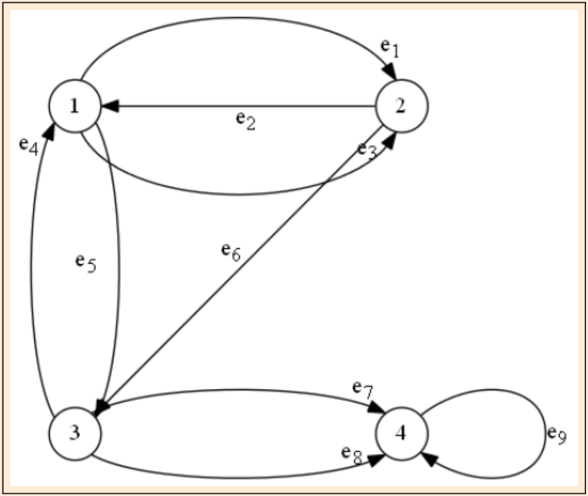
$$\mathbf{A}_G = \begin{bmatrix} 0 & 3 & 2 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\mathbf{B}_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Exercise 12: Matrix Representation for Directed Graph

Exercise

Determine the adjacency matrix and incidence matrix of the following graph G



Solution:

$$\mathbf{A}_G = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}_G = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$