Binomial Coefficients (Supplementary)

Discrete Mathematics – Second Term 2022-2023

MZI

School of Computing
Telkom University

SoC Tel-U

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition., 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

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Binomial Coefficient: Motivation

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(x+y)^3 &=
\end{aligned}
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what is the general expression for $(x+y)^n$ for $n \geq 2$?

The Idea for Coefficients Expansion

To find the expression for $(x+y)^n$ we start from $(x+y)^4$. We see that

$$(x+y)^4 = \underbrace{(x+y)}_{S_1}\underbrace{(x+y)}_{S_2}\underbrace{(x+y)}_{S_3}\underbrace{(x+y)}_{S_4}$$

A summand of $(x+y)^4$ must contain:

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To find the expression for $(x+y)^n$ we start from $(x+y)^4$. We see that

$$(x+y)^4 = \underbrace{(x+y)}_{S_1}\underbrace{(x+y)}_{S_2}\underbrace{(x+y)}_{S_3}\underbrace{(x+y)}_{S_4}$$

A summand of $(x+y)^4$ must contain:

- one variable x or y from S_1, but not both,
- one variable x or y from S_2, but not both,
- one variable x or y from S_3, but not both, and
- one variable x or y from S_4, but not both.

We can write the expansion of $(x+y)^4$ as

$$(x+y)^4 \quad =$$

We can write the expansion of $(x+y)^4$ as

$$\begin{aligned}(x+y)^4 &= xxxx + xxxy + xxyx + xxyy + xyxx + xyxy + xyyx + xyyy \\ &\quad + yxxx + yxxy + yxyx + yxyy + yyxx + yyxy + yyyx + yyyy.\end{aligned}$$

All the summands must be of the form $r_1 r_2 r_3 r_4$ where $r_i \in \{x, y\}$ for $1 \leq i \leq 4$. Observe that:

- The coefficient of x^4 can be obtained if exactly 0 of r_1, r_2, r_3, r_4 equals to y.

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- The coefficient of x^4 can be obtained if exactly 0 of r_1, r_2, r_3, r_4 equals to y. This can be done in $\binom{4}{0}$ ways. Similarly, it can be obtained if exactly 4 of r_1, r_2, r_3, r_4 equal to x, so, we have $\binom{4}{4}$ ways to do so.
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Contents

The Binomial Theorem

Theorem

For every $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} & (x+y)^n \\ = & \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \sum_{j=0}^{n} \binom{n}{n-j} x^{n-j} y^j \\ = & \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n. \end{aligned}$$

Proof (Proof can be also obtained using induction)

The equation is obviously true for $n = 1$.

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The equation is obviously true for $n = 1$. Now, assume that $n \geq 2$.

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Proof (Proof can be also obtained using induction)

The equation is obviously true for $n = 1$. Now, assume that $n \geq 2$. Without using commutativity and associativity of real numbers arithmetic, every summand in the expansion can be written as $r_1 r_2 \cdots r_n$ with $r_i \in \{x, y\}$, $\forall i \, (1 \leq i \leq n)$.

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Contents

Pascal Triangle and Pascal's Identity

In high school we have learned that the expansion of $(x+y)^n$ can be obtained by using Pascal's triangle below.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\cdots$$

By Pascal's identity:

$$\binom{6}{4}+\binom{6}{5}=\binom{7}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
...

Pascal's Identity

Theorem

Given two positive integers n and k with $n > k$, then $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

Proof

The proof is left as one of the *challenging problems at the end of this slide*.

Contents

Exercise

Exercise

1. Find the coefficient of x^2y^3 in the expansion of $(x - y)^5$.
2. Find the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$.
3. Find the coefficient of x^{18} in the expansion of $\left(x + \frac{1}{x}\right)^{100}$.

Solution: No. 1 and No. 2

1. Observe that $(x-y)^5 =$

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1. Observe that $(x-y)^5 = (x+(-y))^5 = \sum_{j=0}^{5} \binom{5}{j} (x)^j (-y)^{5-j}$. The coefficient of x^2y^3 can be obtained if $j =$

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$$\binom{5}{2} \cdot (x)^2 (-y)^{5-2} =$$

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$$\binom{5}{2} \cdot (x)^2 (-y)^{5-2} = \binom{5}{2} (x)^2 (-y)^3 =$$

Solution: No. 1 and No. 2

1. Observe that $(x-y)^5 = (x+(-y))^5 = \sum_{j=0}^{5} \binom{5}{j} (x)^j (-y)^{5-j}$. The coefficient of x^2y^3 can be obtained if $j=2$, so the summand in the form of x^2y^3 is

$$\binom{5}{2} \cdot (x)^2 (-y)^{5-2} = \binom{5}{2} (x)^2 (-y)^3 = -\binom{5}{2} x^2 y^3.$$

   Thus, the coefficient of x^2y^3 is $-\binom{5}{2} = -10$.

2. Observe that $(2x-3y)^{25} =$

Solution: No. 1 and No. 2

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Contents

Challenging Problems

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1. A closed form of a mathematical expression is an expression that does not include sigma notation or equivalent sums of terms that change depending on the domain. For example, we are familiar with these expressions:
   - $1+2+3+\cdots+n=\sum_{i=1}^{n} i=\frac{(n)(n+1)}{2}$, the form $\frac{(n)(n+1)}{2}$ is the closed form of $\sum_{i=1}^{n} i$,
   - $1+\frac{1}{2}+\frac{1}{4}+\cdots=\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i}=\frac{1}{1-\frac{1}{2}}=2$, the value 2 is the closed form of $\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i}$.

   Find the closed form of

   1. $\sum_{j=0}^{n}\binom{n}{j}$
   2. $\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}$
   3. $\sum_{j=0}^{n}\binom{n}{j}2^{j}$
2. Prove the Pascal's identity.