Generalized Permutation and Combination

Discrete Mathematics – Second Term 2022-2023

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Course Material

1. Permutation with Repetition
2. Permutation with Identical Object
3. Combination with Repetition
4. Challenging Problems
5. Summary of Formula

Contents

Permutation with Repetition

Definition (Permutation with Repetition)

Permutation with repetition is a permutation that allows similar objects occur more than once.

Example

How many string of length 10 that can be composed **only from capital letter** if each capital letter can be used more than once?

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Solution: Suppose the string is $s_1 s_2 \cdots s_{10}$. There are 26 capital letters and each letter can be reused, then there are 26 possibilities for each character s_i $(1 \leq i \leq 10)$. Based on the product rule, there are 26^{10} strings of length 10 that can be composed from capital letters.

Theorem (The number of Permutation with Repetition)

The number of r-permutation of a set with n elements that allows repetition is n^r.

Proof

Obvious (by using the product rule). □

Contents

Permutation with Identical Object

Definition

Permutation with identical object is a permutation of n objects that consists of exactly k types of identical object.

Theorem

Given a collection of objects that consists of k types, if each type contain $n_1, n_2, \cdots, n_k$ identical objects, respectively, then the number of different permutations of the objects is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!},$$

sometimes the above expression is written as $P\left(n; n_1, \ldots, n_k\right)$.

Proof

The proof can be obtained easily through the product rule and the division rule.

Problem 1

Exercise

1. Determine the number of different strings that can be composed from all of the letters in the word **ZOOKEEPER** if all of the letters in the string must be used.
2. Determine the number of different strings that can be composed from all of the letters in the word **MISSISSIPPI** if all of the letters in the string must be used.
3. On the shelf, there are 8 books consisting of 4 algorithm books, 2 discrete math books, and 2 statistics books. Determine the number of different ways to arrange these books on the shelf.

Solution of Problem 1

1. Suppose all of the characters in **ZOOKEEPER** are distinguished, namely $\mathbf{ZO}_1\mathbf{O}_2\mathbf{KE}_1\mathbf{E}_2\mathbf{PE}_3\mathbf{R}$, then there are 9 different characters.

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2. Suppose all of the characters in **MISSISSIPPI** are distinguished, namely $\mathbf{MI}_1\mathbf{S}_1\mathbf{S}_2\mathbf{I}_2\mathbf{S}_3\mathbf{S}_4\mathbf{I}_3\mathbf{P}_1\mathbf{P}_2\mathbf{I}_4$, then there are 11 different characters.

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3. Suppose the algorithm books are denoted with $\mathbf{A}$, discrete math books are denoted with $\mathbf{M}$, and statistics books are denoted with $\mathbf{S}$. If all books are distinguished, then the books arrangement on the shelf can be viewed as a permutation of the string $\mathbf{A_1A_2A_3A_4M_1M_2S_1S_2}$.

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Contents

Motivation: Combination with Repetition

- We have already seen that for a set A of n elements there are $\binom{n}{k}$ different ways to construct the subsets of A containing exactly k elements.
- In the above case, the repetition is not allowed, which means that if an element x has been taken from A, then x **cannot be taken again** from A. Now, what if the repetition is allowed?

Example of Combination with Repetition Case

Example

You walk to a mart that sells three type of fruits, namely apple, orange, and melon. How many ways to choose four fruits that contain a combination of these three fruits (regardless of the order of taking and assume that each fruit of the same type is identical)?

Solution (*brute force version*):

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- 3 fruits of the same type, 1 different fruit:

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So there are $3 + 6 + 3 + 3 = 15$ ways to choose four fruits that contain a combination of the three fruits.

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If we consider the order of each object (namely 4 letters of b and 2 signs of $-$) there are $\underbrace{4}_{\#\text{ fruits}} + \underbrace{2}_{\#\text{ sign }-} = 6$ objects whose positions can be switched.

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Based on the division rule, the number of ways to place the 6 objects without distinguishing the letter b and the sign $-$ is

If we consider the order of each object (namely 4 letters of b and 2 signs of $-$) there are $\underbrace{4}_{\#\text{ fruits}} + \underbrace{2}_{\#\text{ sign } -} = 6$ objects whose positions can be switched. We obtain $6!$ ways of object placements if we distinguished the letter b and the sign $-$. Notice that:

- there are 2 identical $-$ signs (there are $2!$ similar ways of placement for $-$)
- there are 4 identical b letters (there are $4!$ similar ways of placement for b).

Based on the division rule, the number of ways to place the 6 objects without distinguishing the letter b and the sign $-$ is $\frac{6!}{4! \cdot 2!} = 15$ ways.

Combination with Repetition

Definition (Combination with Repetition)

An r-combination with repetition is the number of ways to take r objects from n **types** of different objects that allows each type of object is taken until r times.

Example

Determine the number of solution for the equation $x + y + z = 11$ with the requirement $x, y, z \in \mathbb{N}_0$.

Example

For example, these are some values of x, y, and z that satisfies the equation:

- $x = 0$, $y = 0$, $z = 11$,
- $x = 0$, $y = 1$, $z = 10$,
- $x = 1$, $y = 0$, $z = 10$,
- $\vdots$
- $x = 4$, $y = 4$, $z = 3$, etc..

Solution: Notice that the possible values of x, y, z is $0 \leq x, y, z \leq 11$ and $x, y, z \in \mathbb{N}_0$.

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Solution: Notice that the possible values of x, y, z is $0 \leq x, y, z \leq 11$ and $x, y, z \in \mathbb{N}_0$. Suppose the value of each x, y, z is represented by the number 1 and for each different variable, the number 1 is separated by $-$. For example:

- $1111 - 1111 - 111$ denotes the value of $x =$

Solution: Notice that the possible values of x, y, z is $0 \leq x, y, z \leq 11$ and $x, y, z \in \mathbb{N}_0$. Suppose the value of each x, y, z is represented by the number 1 and for each different variable, the number 1 is separated by $-$. For example:

- $1111 - 1111 - 111$ denotes the value of $x = 4$, value of $y =$

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- $1111 - 1111 - 111$ denotes the value of $x = 4$, value of $y = 4$, and value of $z = 3$.
- $11 - 111 - 111111$ denotes the value of $x =$

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- $11 - 111 - 111111$ denotes the value of $x = 2$, value of $y =$

Solution: Notice that the possible values of x, y, z is $0 \leq x, y, z \leq 11$ and $x, y, z \in \mathbb{N}_0$. Suppose the value of each x, y, z is represented by the number 1 and for each different variable, the number 1 is separated by $-$. For example:

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- $11 - 111 - 111111$ denotes the value of $x = 2$, value of $y = 3$, and value of $z = 6$

The number of solutions is the number of different permutation of the string $\underbrace{111 \cdots 1}_{\text{11 number 1}} - -$. Using the product rule and the division rule, the number of different permutations is

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The number of solutions is the number of different permutation of the string $\underbrace{111\cdots1}_{\text{11 number 1}} - -$. Using the product rule and the division rule, the number of different permutations is

$$\frac{(11+2)!}{11! \cdot 2!} = 78$$

So there are 78 different solutions for the equation $x + y + z = 11$ with the requirement $x, y, z \in \mathbb{N}_0$.

Problem 2

Exercise

1. Determine the number of solutions for the equation $x_1 + x_2 + \cdots + x_n = r$ if $x_i \in \mathbb{N}_0$ for all $1 \leq i \leq n$.
2. Determine the number of ways to share **a dozen** of identical donuts into **four** different boxes if:
    1. it is possible to have an empty box,
    2. no empty box

   Requirement: all the donuts must be put to a box (note: 1 dozen $= 12$ donuts).

Solution of Exercise 1 Problem 2

Notice that the possible value of x_i is $0 \leq x_i \leq r$ and $x_i \in \mathbb{N}_0$.

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$$\frac{(\#\text{ number } 1 + \# \quad -\text{sign})!}{(\#\text{ number } 1)!\,(\# \ -\text{sign})!} =$$

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$$\frac{(\#\ \text{number } 1 + \#\ \ - \text{sign})!}{(\#\ \text{number } 1)!\,(\#\ \ -\text{sign})!} = \frac{(r+n-1)!}{r! \cdot (n-1)!}$$

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We can conclude that the number of solutions for the equation
$x_1 + x_2 \cdots + x_n = r$ is

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$$\frac{(12+3)!}{12! \cdot 3!} = 455$$

So there are 455 ways to distribute a dozen donuts to four boxes.

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The number of ways to distribute the donut can also be modeled as the number of solutions to the equation

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$$x_1 + x_2 + x_3 + x_4 = 12,\ x_i \in \mathbb{N}_0 \text{ for every } 1 \leq i \leq 4.$$

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$$\frac{(8+3)!}{8! \cdot 3!} = 165,$$

so there are 165 ways to distribute a dozen donuts.

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Notes

The number of ways to distribute the donut can also be modeled as the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 12,\ x_i \geq 1 \text{ for every } 1 \leq i \leq 4.$$

Formula of r-Combination with Repetition

Theorem

The number of r-combination with repetition is the number of ways to take r objects from n **types** of objects that allows every type of object is taken more than once is

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}.$$

Proof

Left to the reader as an exercise.

You can remember the above theorem as follows: the value of r-combination with repetition from n types of different objects is

$$\binom{(\#\text{ taken}) + (\#\text{ type}) - 1}{\#\text{ taken}} = \binom{(\#\text{ taken}) + (\#\text{ type}) - 1}{(\#\text{ type }) - 1}$$

Contents

Challenging Problems

Challenging Problems

1. A farmer has five identical goats that can be placed in three different cages. If each cage cannot be empty, how many different ways to place the goats are there?
2. We know that the number of different solutions to the equation $\sum_{i=1}^{n} x_i = r$ with the constraint $\forall i \, (x_i \in \mathbb{N}_0)$ is $\binom{r+n-1}{r}$. Determine the number of non-negative integers solutions from the following equations
    1. $x_1 + x_2 + x_3 = 11$ with $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$.
    2. $x_1 + x_2 + x_3 = 11$ with $0 \leq x_1 \leq 2$ and $x_2 > 1$.

Contents

Summary: Formula of Permutation and Combination with Repetition

	r-permutation	r-combination
with repetition	n^r	$\binom{r+n-1}{r}$
without repetition	$P(n,r)$	$\binom{n}{r}$

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There are other types of permutation and combination. Basically, all of the formula of generalized permutation and combination can be obtained using solely **four** basic counting techniques, namely **the sum rule, the product rule, inclusion-exclusion principle (the subtraction rule), and the division rule**. Please read the textbook for details.