Permutation and Combination

Discrete Mathematics – Second Term 2022-2023

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Course Material

Contents

Permutation

Sometimes we need to count how many ways to arrange particular objects regarding their order.

Definition

A permutation of a set of objects is an ordered arrangement of these objects.

Definition (r-*Permutation*)

An r-permutation is an ordered arrangement of r objects.

Permutation on a Set

Example

Suppose $A = \{1, 2, 3\}$.

1. Each of $(3, 1, 2)$, $(2, 3, 1)$, and $(1, 3, 2)$ is a permutation of the elements of A. Here $(3, 1, 2)$, $(2, 3, 1)$, and $(1, 2, 3)$ are different.
2. Each of $(3, 1)$ and $(1, 3)$ is a 2-*permutation* of the elements of A. Here $(3, 1)$ and $(1, 3)$ are different.

Permutation Notation

An r-permutation of a set of n elements is denoted as

$$P(n, r) \text{ or } P_r^n \text{ or } {}_nP_r \text{ or } {}^nP_r \text{ or } P_{n,r}.$$

Permutation and Product Rule

Theorem

If $0 \leq r \leq n$ then $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$.

Proof

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Proof

Notice that there are $(n-i+1)$ ways to choose the i-th element of permutation (remember the problem about the number of dancing pairs on the Basic Counting Technique slide). Because there are r elements, then according to the product rule $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$. □

Factorial Notation

Definition

For all integers $n \geq 0$, we define $n!$ recursively as follows

$$n! = \begin{cases} n \cdot (n-1)! & \text{, if } n \geq 0 \\ 1 & \text{, if } n = 0. \end{cases}$$

Using this notation, we have

$$\begin{aligned} P(n,r) &= \frac{n(n-1)\cdots(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \\ P(n,n) &= \frac{n!}{(n-n)!} = n! \end{aligned}$$

$n!$ denotes the number of different ways to arrange n objects.

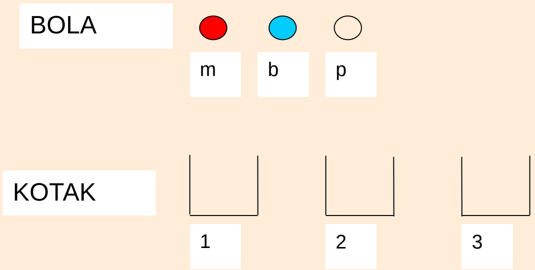
$0! = 1$

$0!$ denotes the number of different ways to arrange 0 objects. Notice that there is 1 way to arrange 0 object.

Simple Permutation

Example

There are three balls whose colors are red, blue, and white. All of them will be put in one of the three boxes labelled as 1, 2, and 3, respectively. If each box can only contain one ball, how many different ways to put these three balls?



Suppose the red, blue, and white balls are denoted with r, b, and w, respectively. We can construct the following table to write all of possible enumerations.

Box 1	Box 2	Box 3	Order
r	b	w	rbw

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By observation, the placement problem of the three balls is a 3-permutation of 3 objects, namely $P(3,3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = (3)(2)(1) = 6$. This problem can also be solved using the product rule.

Example of Permutation Problem

Example of Permutation Problem

1. Three exams, namely discrete mathematics, calculus, and matrices and vector spaces will be conducted in a day among five weekdays (Monday until Friday). If there is no exam that is conducted in the same day, how many ways to arrange different exam schedules?
2. How many permutations of a string **EQUATORS** that contain a string "**EQU**"?
   (Example: **EQUATORS, AEQUTORS**, **STORAEQU**, **SEQUATOR**, **SORTEQUA**, **TROASEQU**, etc..)
3. How many permutations of a string **EQUATORS** that contain the string "**EQU**" and "**TOR**"?
   (Example: **EQUATORS**, **AEQUTORS**, **STORAEQU**, **SEQUATOR**, and **TORASEQU**; however, **SORTEQUA**, **TROASEQU**, **TORQUASE**, and **QUASETOR** are not included in this kind of string because there is no string "EQU" and "TOR" together).

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3. We consider the string **EQU** as a block and denote this block by **X**. We also consider the string **TOR** as a block and denote the block with **Y**. The problem becomes the number of permutation of four objects, namely **X**,**A**,**Y**,**S**. Therefore, there are

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3. We consider the string **EQU** as a block and denote this block by **X**. We also consider the string **TOR** as a block and denote the block with **Y**. The problem becomes the number of permutation of four objects, namely **X**,**A**,**Y**,**S**. Therefore, there are $4! = 24$ permutations of string **EQUATORS** that contain string **EQU** and **TOR**.

Contents

Combination

Sometimes we need to count the number of ways to arrange particular object regardless their order.

Definition (Combination)

A combination of a set of objects is an unordered arrangement of these objects.

Definition (r-*combination*)

An r-combination is an unordered arrangement of r objects.

Example

Suppose $A = \{1, 2, 3, 4, 5, 6\}$.

1. $\{1, 5, 6\}$ is a 3-combination of elements of A. Here $\{1, 5, 6\}$, $\{1, 6, 5\}$, $\{5, 1, 6\}$, $\{5, 6, 1\}$, $\{6, 1, 5\}$, $\{6, 5, 1\}$ are considered as identical 3-combination (the order does not matter).
2. $\{4, 5\}$ is a 2-combination of elements of A. There are 15 different 2-combinations of elements of A (try by yourself!).

Combination Notation

An r-combination of a set with n elements is denoted as

$$C(n, r) \text{ or } C_r^n \text{ or } {}_nC_r \text{ or } {}^nC_r \text{ or } C_{n,r} \text{ or } \binom{n}{r}.$$

$\binom{n}{r}$ is also called as binomial coefficient and read as: "n taken r" or "n choose r".

Combination, Product Rule, and Division Rule

Theorem

If $0 \leq r \leq n$ then $C(n, r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)}$.

Proof

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Regarding the order, there are $P(n,r)$ ways to choose r objects from n objects. Afterwards, because the order does not matter, then based on the division rule there are

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Proof

Regarding the order, there are $P(n, r)$ ways to choose r objects from n objects. Afterwards, because the order does not matter, then based on the division rule there are $\frac{P(n,r)}{r!}$ different ways to choose r objects of n objects (remember the problem on Basic Counting Technique slide about the number of subsets with cardinality k of set A where $|A| = n$).

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If $0 \leq r \leq n$ then $C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)}$.

Proof

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Corollary

$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{(n-r)!r!}$.

Combination Identity

Corollary

$C(n,r) = C(n,n-r)$, or we can write $\binom{n}{r} = \binom{n}{n-r}$

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Notice that

$$C(n,r) = \frac{n!}{(n-r)!r!} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-(n-r))!(n-r)!} = C(n, n-r).$$

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□

Proof (Combinatorial Proof)

Given n objects, choosing r objects from n objects (namely $\binom{n}{r}$) is **identical** with leaving $n-r$ objects from n objects (namely $\binom{n}{n-r}$). □

Simple Combination

Example

There are two identical yellow balls that will be put in the three boxes labeled as 1, 2, and 3. If each box cannot contain more than one ball, how many different ways of ball placement are there?

Suppose the balls are differentiate first, these balls are denoted by a and b. Ball a can be put in

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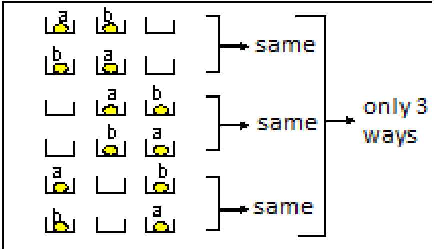

Example of Combination Problem

Example of Combination Problem

1. A football club consists of 23 players. How many different *starting lineup* that can be composed if each player can play in any position? (Note: *starting lineup* is the eleven players that reside on the pitch at the start of the game).
2. How many binary strings of length 8 that contain exactly 3 digits of 1? (Example: 11100000, 0111000, 01010100, 11000010, 10100100, etc..)
3. In a class, there are 25 male students and 20 female students. How many ways to form a class representative containing four students such that at least one of them is a male and at least one of them is a female.

Solution of Combination Problem Example (1 & 2)

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$$s_1 \; s_2 \; s_3 \; s_4 \; s_5 \; s_6 \; s_7 \; s_8.$$

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   If we notice all of the possible positions of character 1 that are allowed, then the possible position form a 3-combination of $\{1, 2, 3, \ldots, 8\}$. So there are $\binom{8}{3} = 56$ different binary strings of length 8 that exactly contain three digits of 1.

Solution of Combination Problem Example (3, version 1)

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   - If a delegation consists of 1 male student and 3 female students, then there are $\binom{25}{1}$ ways to choose male students and $\binom{20}{3}$ ways to choose female students. Using the product rule, this can be done in $\binom{25}{1}\binom{20}{3}$ ways.

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Solution of Combination Problem Example (3, version 1)

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   - If a delegation consists of 1 male student and 3 female students, then there are $\binom{25}{1}$ ways to choose male students and $\binom{20}{3}$ ways to choose female students. Using the product rule, this can be done in $\binom{25}{1}\binom{20}{3}$ ways.
   - If a delegation consists of 2 male students and 2 female students, then there are $\binom{25}{2}$ ways to choose male students and $\binom{20}{2}$ ways to choose female students. Using the product rule, this can be done in $\binom{25}{2}\binom{20}{2}$ ways.
   - If a delegation consists of 3 male students and 1 female student, then there are $\binom{25}{3}$ ways to choose male students and $\binom{20}{1}$ ways to choose female students.

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   These three cases are independent, so using the sum rule, the construction of delegation can be done in $\binom{25}{1}\binom{20}{3}+\binom{25}{2}\binom{20}{2}+\binom{25}{3}\binom{20}{1}=131\,500$ ways.

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   These three cases are independent, so using the sum rule, the construction of delegation can be done in $\binom{25}{1}\binom{20}{3}+\binom{25}{2}\binom{20}{2}+\binom{25}{3}\binom{20}{1}=131\,500$ ways. We CANNOT obtain this result from $\binom{25}{1}\binom{20}{1}\binom{43}{2}$, because $\binom{43}{2}$ DOES NOT distinguish the taking of male and female students. (Furthermore, $\binom{25}{1}\binom{20}{1}\binom{43}{2}=451\,500$.)

Solution of Combination Problem Example (3, version 2)

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- Q : # ways to compose a delegation of 4 people
- R : # ways to compose a delegation where all four of them are male students
- S : # ways to compose a delegation where all four of them are female students

Therefore $P = \binom{45}{4} - \binom{25}{4} - \binom{20}{4} = 131\,500$.

Contents

Problem 1

Exercise

1. Sheldon, Leonard, Howard, Rajesh, and Stuart go to watch a movie. When they arrive at the theatre, there are only seven seats at the front row remaining. How many different ways of seating if **Howard and Rajesh must not sit side by side**?
2. A football club has 23 players and 3 of them are goalkeepers. How many different *starting lineups* can be composed if:
   1. exactly one goalkeeper must play (regardless the formation that is used),
   2. the football club can use the formation 4-4-2 or 4-3-3, and all of the players (excluding the goalkeeper) can play as defender, midfielder, or striker. (In football, there is only one goalkeeper for each team that can play on the pitch at a game.)

Solution of Exercise 1 Problem 1

Suppose R : $\#$ the ways of seating if Howard and Rajesh must not sit side by side.

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- Howard sit at the right side of Rajesh, here Howard and Rajesh are viewed as one object. The number of ways to sit is a 4-permutation of 6 objects (when Howard and Rajesh is regarded as one object, then both of them are placed next to each other) or $P(6,4)$.

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Based on the sum rule, the number of ways to sit with the requirement that Howard and Rajesh are sitting side by side is $2 \cdot P(6,4)$. Therefore,

$$R = P(7,5) - 2 \cdot P(6,4) = \frac{7!}{(7-5)!} - 2 \cdot \frac{6!}{(6-4)!} = 1800.$$

Solution of Exercise 2 Problem 1

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For the second problem, suppose D, M, S respectively denotes the ways to choose defenders, midfielders, and strikers of 20 non-goalkeeper players. Notice two following cases:

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Based on the sum rule, the number of different *starting lineups* is $\binom{3}{1}\binom{20}{4}\binom{16}{4}\binom{12}{2} + \binom{3}{1}\binom{20}{4}\binom{16}{3}\binom{13}{3} = 4\,073\,869\,800$ (this arrangement is greater than the first problem because in the first problem the formation is ignored).

Problem 2

Exercise

A university wants to send a team of five students to a competition. From the first selection phase, ten students are selected. Two of them are Alice and Bob. Determine the number of ways to select these five students if:

1. Alice must be chosen;
2. Alice must not be chosen;
3. Alice must be chosen, but Bob must not be chosen;
4. Alice and Bob must be chosen;
5. Both Alice and Bob must not be chosen (the team cannot contain Alice or Bob);
6. Alice or Bob must be chosen (the team must contain Alice or Bob or both of them);
7. Alice or Bob must be chosen, but not both of them.

Solution of Problem 2

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2. Because Alice must not be chosen, then we choose 5 people from 9 people (i.e., 10 people without Alice). The number of ways of choosing is $\binom{9}{5} = 126$.
3. Because Alice must be chosen and Bob must not be chosen, then

Solution of Problem 2

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3. Because Alice must be chosen and Bob must not be chosen, then we choose 4 people from 8 people (i.e., 10 people minus Alice and Bob). The number of ways of choosing is

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3. Because Alice must be chosen and Bob must not be chosen, then we choose 4 people from 8 people (i.e., 10 people minus Alice and Bob). The number of ways of choosing is $\binom{8}{4} = 70$.
4. Because both Alice and Bob must be chosen, then

Solution of Problem 2

1. Because Alice must be chosen, then we only choose 4 people from 9 people (i.e., 10 people without Alice). The number of ways of choosing is $\binom{9}{4} = 126$.
2. Because Alice must not be chosen, then we choose 5 people from 9 people (i.e., 10 people without Alice). The number of ways of choosing is $\binom{9}{5} = 126$.
3. Because Alice must be chosen and Bob must not be chosen, then we choose 4 people from 8 people (i.e., 10 people minus Alice and Bob). The number of ways of choosing is $\binom{8}{4} = 70$.
4. Because both Alice and Bob must be chosen, then we only chose 3 people from 8 people (i.e., 10 people minus Alice and Bob). The number of ways of choosing is

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6. Suppose

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$$|A \cup B| = |A| + |B| - |A \cap B| = \binom{9}{4} + \binom{9}{4} - \binom{8}{3} = 196$$

7. Similar to the previous problem, the number of ways to choose a team that must contain Alice or Bob (but not both of them) is

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$$\begin{aligned} |A \oplus B| &= |A \cup B| - |A \cap B| = |A| + |B| - 2\,|A \cap B| \\ &= \end{aligned}$$

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$$\begin{aligned} |A \oplus B| &= |A \cup B| - |A \cap B| = |A| + |B| - 2\,|A \cap B| \\ &= \binom{9}{4} + \binom{9}{4} - 2\binom{8}{3} = 140. \end{aligned}$$

Contents

Challenging Problems

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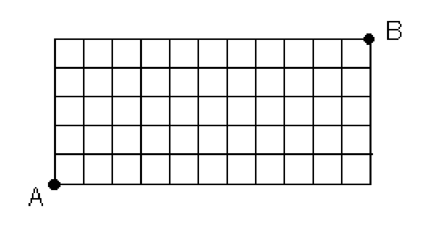
1. If there are four female students and four male students, how many different seating arrangements with alternating female and male students in a row that consists of eight seats?
2. Determine the **number of different shortest paths** from A to B in the following diagram:



At a time, the allowed movement is only one unit horizontally or vertically.