Predicate Logic 2: Truth of Formulas with Single Quantifier – Negation of Quantified Formulas

Mathematical Logic – First Term 2023-2024

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October 2023

Acknowledgements

This slide is compiled using the materials in the following sources:

1. *Discrete Mathematics and Its Applications* (Chapter 1), 8th Edition, 2019, by K. H. Rosen (primary reference).
2. *Discrete Mathematics with Applications* (Chapter 3), 5th Edition, 2018, by S. S. Epp.
3. *Logic in Computer Science: Modelling and Reasoning about Systems* (Chapter 2), 2nd Edition, 2004, by M. Huth and M. Ryan.
4. *Mathematical Logic for Computer Science* (Chapter 5, 6), 2nd Edition, 2000, by M. Ben-Ari.
5. Discrete Mathematics 1 (2012) slides in Fasilkom UI by B. H. Widjaja.
6. Mathematical Logic slides in Telkom University by A. Rakhmatsyah and B. Purnama.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to `<pleasedontspam>@telkomuniversity.ac.id`.

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Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then

$$\forall x\ P(x) \quad \equiv$$

Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then

$$\begin{aligned}\forall x\ P(x) &\equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \\ &\equiv \end{aligned}$$

Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then

$$\begin{aligned} \forall x\ P(x) &\equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \\ &\equiv \left(0^2 < 10\right) \wedge \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \\ &\equiv \end{aligned}$$

Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
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Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then

$$\begin{aligned}\forall x\ P(x) &\equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \\ &\equiv \left(0^2 < 10\right) \wedge \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \\ &\equiv (0 < 10) \wedge (1 < 10) \wedge (4 < 10) \wedge (9 < 10) \equiv\end{aligned}$$

Truth of Formulas with Single Quantifier (1)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 < 10$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then

$$\begin{aligned} \forall x\ P(x) &\equiv P(0) \wedge P(1) \wedge P(2) \wedge P(3) \\ &\equiv \left(0^2 < 10\right) \wedge \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \\ &\equiv (0 < 10) \wedge (1 < 10) \wedge (4 < 10) \wedge (9 < 10) \equiv \mathrm{T}. \end{aligned}$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$\forall x \; P(x) \quad \equiv$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$
\begin{aligned}
\forall x\ P(x) &\equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\
&\equiv
\end{aligned}
$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$\begin{aligned} \forall x\ P(x) &\equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\ &\equiv \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \wedge \left(4^2 < 10\right) \\ &\equiv \end{aligned}$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$\begin{aligned}
\forall x\ P(x) &\equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\
&\equiv \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \wedge \left(4^2 < 10\right) \\
&\equiv (1 < 10) \wedge (4 < 10) \wedge (9 < 10) \wedge (16 < 10) \equiv
\end{aligned}$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$\begin{aligned}
\forall x\ P(x) &\equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\
&\equiv \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \wedge \left(4^2 < 10\right) \\
&\equiv (1 < 10) \wedge (4 < 10) \wedge (9 < 10) \wedge (16 < 10) \equiv \mathrm{F}.
\end{aligned}$$

2. If the domain is $\{1, 2, 3, 4\}$, then

$$\begin{aligned}
\forall x \; P(x) &\equiv P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\
&\equiv \left(1^2 < 10\right) \wedge \left(2^2 < 10\right) \wedge \left(3^2 < 10\right) \wedge \left(4^2 < 10\right) \\
&\equiv (1 < 10) \wedge (4 < 10) \wedge (9 < 10) \wedge (16 < 10) \equiv \mathrm{F}.
\end{aligned}$$

In this case, 4 is the *counterexample* of the formula $\forall x \; \left(x^2 < 10\right)$ over the domain $\{1, 2, 3, 4\}$.

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x \; P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x \; P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x \; P(x) \equiv$

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv$

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x \; P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x \; P(x)$ if the domain is:

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Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x \; P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv$ $\left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv$

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
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Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv$ $\left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv$

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

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3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$,

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false.

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$.

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
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Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$. In this case $x = \frac{1}{2}$ is the *counterexample* of $\forall x\ \left(x^2 \geq x\right)$ over the domain $\mathbb{R}$.

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$. In this case $x = \frac{1}{2}$ is the *counterexample* of $\forall x\ \left(x^2 \geq x\right)$ over the domain $\mathbb{R}$.
3. If $x \geq 1$,

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv$ $\left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$. In this case $x = \frac{1}{2}$ is the *counterexample* of $\forall x\ \left(x^2 \geq x\right)$ over the domain $\mathbb{R}$.
3. If $x \geq 1$, multiplying both sides with x implies

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$. In this case $x = \frac{1}{2}$ is the *counterexample* of $\forall x\ \left(x^2 \geq x\right)$ over the domain $\mathbb{R}$.
3. If $x \geq 1$, multiplying both sides with x implies $x^2 \geq x$, so $x^2 \geq x$ is true. Therefore

Truth of Formulas with Single Quantifier (2)

Exercise

Let $\forall x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 \geq x$". Determine the truth value of $\forall x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2\}$
2. the set of real numbers $\mathbb{R}$
3. the set $\{1, 2, 3, \ldots\}$

Solution:

1. If the domain is $\{0, 1, 2\}$, then $\forall x\ P(x) \equiv P(0) \wedge P(1) \wedge P(2) \equiv \left(0^2 \geq 0\right) \wedge \left(1^2 \geq 1\right) \wedge \left(2^2 \geq 2\right) \equiv (0 \geq 0) \wedge (1 \geq 1) \wedge (4 \geq 2) \equiv \mathrm{T}$.
2. If the domain is the set of real numbers $\mathbb{R}$, for $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2} = x$, or in other words $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is false. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{F}$. In this case $x = \frac{1}{2}$ is the *counterexample* of $\forall x\ \left(x^2 \geq x\right)$ over the domain $\mathbb{R}$.
3. If $x \geq 1$, multiplying both sides with x implies $x^2 \geq x$, so $x^2 \geq x$ is true. Therefore $\forall x\ P(x) \equiv \forall x\ \left(x^2 \geq x\right) \equiv \mathrm{T}$.

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x \ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x \ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
   $\exists x\ P(x) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
   $\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then $\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv$

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Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
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Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
$\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv$

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Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
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Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then $\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.

Truth of Formulas with Single Quantifier (3)

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Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
   $\exists x \; P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.
2. If the domain is $\{1, 2, 3, 4\}$, then
   $\exists x \; P(x) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
$\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.
2. If the domain is $\{1, 2, 3, 4\}$, then
$\exists x\ P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then $\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.
2. If the domain is $\{1, 2, 3, 4\}$, then $\exists x\ P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \equiv \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \vee \left(4^2 > 10\right) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
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Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
$\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.
2. If the domain is $\{1, 2, 3, 4\}$, then
$\exists x\ P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \equiv \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \vee \left(4^2 > 10\right) \equiv (1 > 10) \vee (4 > 10) \vee (9 > 10) \vee (16 > 10) \equiv$

Truth of Formulas with Single Quantifier (3)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$x^2 > 10$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{0, 1, 2, 3\}$
2. the set $\{1, 2, 3, 4\}$

Solution:

1. If the domain is $\{0, 1, 2, 3\}$, then
$\exists x\ P(x) \equiv P(0) \vee P(1) \vee P(2) \vee P(3) \equiv \left(0^2 > 10\right) \vee \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \equiv (0 > 10) \vee (1 > 10) \vee (4 > 10) \vee (9 > 10) \equiv \mathrm{F}$.
2. If the domain is $\{1, 2, 3, 4\}$, then
$\exists x\ P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4) \equiv \left(1^2 > 10\right) \vee \left(2^2 > 10\right) \vee \left(3^2 > 10\right) \vee \left(4^2 > 10\right) \equiv (1 > 10) \vee (4 > 10) \vee (9 > 10) \vee (16 > 10) \equiv \mathrm{T}$.

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have
   $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv$

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have
$\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathbf{F}$.

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have
$\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$,

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathbf{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathbf{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv$

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{Z}$.

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{Z}$.
3. If the domain is $\mathbb{R}$,

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{Z}$.
3. If the domain is $\mathbb{R}$, then we have $1 \in \mathbb{R}$ and

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{Z}$.
3. If the domain is $\mathbb{R}$, then we have $1 \in \mathbb{R}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv$

Truth of Formulas with Single Quantifier (4)

Exercise

Let $\exists x\ P(x)$ be a formula where $P(x)$ is the statement "$\frac{1}{x} \geq x$". Determine the truth value of $\exists x\ P(x)$ if the domain is:

1. the set $\{2, 3, 4\}$
2. the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
3. the set of real numbers $\mathbb{R}$

Solution:

1. If the domain is $D = \{2, 3, 4\}$, then we have $\exists x\ P(x) \equiv P(2) \vee P(3) \vee P(4) \equiv \left(\frac{1}{2} \geq 2\right) \vee \left(\frac{1}{3} \geq 3\right) \vee \left(\frac{1}{4} \geq 4\right) \equiv \mathrm{F}$.
2. If the domain is $\mathbb{Z}$, then we have $1 \in \mathbb{Z}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{Z}$.
3. If the domain is $\mathbb{R}$, then we have $1 \in \mathbb{R}$ and $\frac{1}{1} \geq 1$. Therefore $\exists x\ P(x) \equiv \mathrm{T}$ over the domain $\mathbb{R}$.

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Truth of Formulas with Two/ More Quantifiers

	true when...	**false** when...
$\forall x \forall y\ P(x,y)$ $\forall y \forall x\ P(x,y)$	$P(x,y)$ **is true** for every pair x, y	$P(x,y)$ **is false** for at least one pair x, y
$\forall x \exists y\ P(x,y)$	For every x, there is a y for which $P(x,y)$ **is true**	There is an x such that $P(x,y)$ **is false** for every y
$\exists x \forall y\ P(x,y)$	There is an x such that $P(x,y)$ **is true** for every y	For every x, there is y for which $P(x,y)$ **is false**
$\exists x \exists y\ P(x,y)$ $\exists y \exists x\ P(x,y)$	$P(x,y)$ **is true** for at least one pair x, y	$P(x,y)$ **is false** for every pair x, y

Recall that $\forall x \exists y\ P(x,y)$ is not equivalent to $\exists y \forall x\ P(x,y)$.

The truth of $\forall x \exists y\ P(x, y)$ and $\exists x \forall y\ P(x, y)$

Suppose $P(x, y)$ is a binary predicate which is evaluated in a domain $D = \{a, b\}$, then

- $\forall x \exists y\ P(x, y) \equiv$

The truth of $\forall x \exists y\ P(x, y)$ and $\exists x \forall y\ P(x, y)$

Suppose $P(x, y)$ is a binary predicate which is evaluated in a domain $D = \{a, b\}$, then

- $\forall x \exists y\ P(x, y) \equiv \forall x\ \ (P(x, a) \vee P(x, b)) \equiv$

The truth of $\forall x \exists y\ P(x,y)$ and $\exists x \forall y\ P(x,y)$

Suppose $P(x,y)$ is a binary predicate which is evaluated in a domain $D = \{a, b\}$, then

- $\forall x \exists y\ P(x,y) \equiv \forall x\ \ (P(x,a) \vee P(x,b)) \equiv$
  $(P(a,a) \vee P(a,b)) \wedge (P(b,a) \vee P(b,b))$
- $\exists x \forall y\ P(x,y) \equiv$

The truth of $\forall x \exists y \; P(x,y)$ and $\exists x \forall y \; P(x,y)$

Suppose $P(x,y)$ is a binary predicate which is evaluated in a domain $D = \{a, b\}$, then

- $\forall x \exists y \; P(x,y) \equiv \forall x \;\; (P(x,a) \vee P(x,b)) \equiv$
  $(P(a,a) \vee P(a,b)) \wedge (P(b,a) \vee P(b,b))$
- $\exists x \forall y \; P(x,y) \equiv \exists x \;\; (P(x,a) \wedge P(x,b)) \equiv$

The truth of $\forall x \exists y\ P(x,y)$ and $\exists x \forall y\ P(x,y)$

Suppose $P(x,y)$ is a binary predicate which is evaluated in a domain $D = \{a, b\}$, then

- $\forall x \exists y\ P(x,y) \equiv \forall x\ \ (P(x,a) \vee P(x,b)) \equiv (P(a,a) \vee P(a,b)) \wedge (P(b,a) \vee P(b,b))$
- $\exists x \forall y\ P(x,y) \equiv \exists x\ \ (P(x,a) \wedge P(x,b)) \equiv (P(a,a) \wedge P(a,b)) \vee (P(b,a) \wedge P(b,b))$

In general, if the domain D is finite, e.g., suppose $D = \{a_1, a_2, \ldots, a_n\}$, then

$$\forall x \exists y \; P(x, y) \quad \equiv$$

In general, if the domain D is finite, e.g., suppose $D = \{a_1, a_2, \ldots, a_n\}$, then

$$
\begin{aligned}
\forall x \exists y\ P(x, y) \quad &\equiv \quad (P(a_1, a_1) \vee P(a_1, a_2) \vee \cdots \vee P(a_1, a_n)) \\
&\quad\ \wedge (P(a_2, a_1) \vee P(a_2, a_2) \vee \cdots \vee P(a_2, a_n)) \\
&\quad\ \wedge \cdots \wedge (P(a_n, a_1) \vee P(a_n, a_2) \vee \cdots \vee P(a_n, a_n)) \\
&\equiv
\end{aligned}
$$

In general, if the domain D is finite, e.g., suppose $D = \{a_1, a_2, \ldots, a_n\}$, then

$$\begin{aligned}
\forall x \exists y\ P(x,y) \quad &\equiv \quad (P(a_1,a_1) \vee P(a_1,a_2) \vee \cdots \vee P(a_1,a_n)) \\
&\quad \wedge (P(a_2,a_1) \vee P(a_2,a_2) \vee \cdots \vee P(a_2,a_n)) \\
&\quad \wedge \cdots \wedge (P(a_n,a_1) \vee P(a_n,a_2) \vee \cdots \vee P(a_n,a_n)) \\
&\equiv \quad \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} P(a_i,a_j)
\end{aligned}$$

and

$$\exists x \forall y\ P(x,y) \quad \equiv$$

In general, if the domain D is finite, e.g., suppose $D = \{a_1, a_2, \ldots, a_n\}$, then

$$\begin{aligned}
\forall x \exists y\ P(x,y) &\equiv (P(a_1,a_1) \vee P(a_1,a_2) \vee \cdots \vee P(a_1,a_n)) \\
&\quad \wedge (P(a_2,a_1) \vee P(a_2,a_2) \vee \cdots \vee P(a_2,a_n)) \\
&\quad \wedge \cdots \wedge (P(a_n,a_1) \vee P(a_n,a_2) \vee \cdots \vee P(a_n,a_n)) \\
&\equiv \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} P(a_i,a_j)
\end{aligned}$$

and

$$\begin{aligned}
\exists x \forall y\ P(x,y) &\equiv (P(a_1,a_1) \wedge P(a_1,a_2) \wedge \cdots \wedge P(a_1,a_n)) \\
&\quad \vee (P(a_2,a_1) \wedge P(a_2,a_2) \wedge \cdots \wedge P(a_2,a_n)) \\
&\quad \vee \cdots \vee (P(a_n,a_1) \wedge P(a_n,a_2) \wedge \cdots \wedge P(a_n,a_n)) \\
&\equiv
\end{aligned}$$

In general, if the domain D is finite, e.g., suppose $D = \{a_1, a_2, \ldots, a_n\}$, then

$$\begin{aligned}
\forall x \exists y\ P(x,y) &\equiv (P(a_1,a_1) \vee P(a_1,a_2) \vee \cdots \vee P(a_1,a_n)) \\
&\quad \wedge (P(a_2,a_1) \vee P(a_2,a_2) \vee \cdots \vee P(a_2,a_n)) \\
&\quad \wedge \cdots \wedge (P(a_n,a_1) \vee P(a_n,a_2) \vee \cdots \vee P(a_n,a_n)) \\
&\equiv \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} P(a_i,a_j)
\end{aligned}$$

and

$$\begin{aligned}
\exists x \forall y\ P(x,y) &\equiv (P(a_1,a_1) \wedge P(a_1,a_2) \wedge \cdots \wedge P(a_1,a_n)) \\
&\quad \vee (P(a_2,a_1) \wedge P(a_2,a_2) \wedge \cdots \wedge P(a_2,a_n)) \\
&\quad \vee \cdots \vee (P(a_n,a_1) \wedge P(a_n,a_2) \wedge \cdots \wedge P(a_n,a_n)) \\
&\equiv \bigvee_{i=1}^{n} \bigwedge_{j=1}^{n} P(a_i,a_j)
\end{aligned}$$

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x \text{ is a student}\}$ and $D_2 = \{y \mid y \text{ is a food}\}$. Predicate $\text{Likes}(x, y)$ means "(student) x likes (food) y". Observe that:

1. $\text{Likes}(Alex, pizza)$ means:

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}\,(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x \text{ is a student}\}$ and $D_2 = \{y \mid y \text{ is a food}\}$. Predicate $\text{Likes}\,(x, y)$ means "(student) x likes (food) y". Observe that:

1. $\text{Likes}\,(Alex, pizza)$ means: "Alex likes pizza"
2. $\forall x\ \text{Likes}\,(x, burger)$ means:

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x$ is a student$\}$ and $D_2 = \{y \mid y$ is a food$\}$. Predicate $\text{Likes}(x, y)$ means "(student) x likes (food) y". Observe that:

1. $\text{Likes}(Alex, pizza)$ means: "Alex likes pizza"
2. $\forall x\ \text{Likes}(x, burger)$ means: "everyone likes burger"
3. $\exists y\ \text{Likes}(Benny, y)$ means:

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}\,(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x \text{ is a student}\}$ and $D_2 = \{y \mid y \text{ is a food}\}$. Predicate $\text{Likes}\,(x, y)$ means "(student) x likes (food) y". Observe that:

1. $\text{Likes}\,(Alex, pizza)$ means: "Alex likes pizza"
2. $\forall x \; \text{Likes}\,(x, burger)$ means: "everyone likes burger"
3. $\exists y \; \text{Likes}\,(Benny, y)$ means: "Benny likes some food"
4. $\forall x \forall y \; \text{Likes}\,(x, y)$ means:

Illustrations of Formulas with Two Quantifiers

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3. $\exists y\ \text{Likes}\,(Benny, y)$ means: "Benny likes some food"
4. $\forall x \forall y\ \text{Likes}\,(x, y)$ means: "everyone likes every food"
5. $\forall x \exists y\ \text{Likes}\,(x, y)$ means:

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}\,(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x \text{ is a student}\}$ and $D_2 = \{y \mid y \text{ is a food}\}$. Predicate $\text{Likes}\,(x, y)$ means "(student) x likes (food) y". Observe that:

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3. $\exists y\ \text{Likes}\,(Benny, y)$ means: "Benny likes some food"
4. $\forall x \forall y\ \text{Likes}\,(x, y)$ means: "everyone likes every food"
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6. $\forall y \exists x\ \text{Likes}\,(x, y)$ means:

Illustrations of Formulas with Two Quantifiers

Suppose $\text{Likes}\,(x, y)$ is a binary predicate over the domain $D_1 \times D_2 = \{(x, y) \mid x \in D_1, y \in D_2\}$, where $D_1 = \{x \mid x \text{ is a student}\}$ and $D_2 = \{y \mid y \text{ is a food}\}$. Predicate $\text{Likes}\,(x, y)$ means "(student) x likes (food) y". Observe that:

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6. $\forall y \exists x\ \text{Likes}\,(x, y)$ means: "for every food, there is someone who likes it" or "every food is liked by someone"
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Determining the truth of quantified formulas

Exercise

Determine the truth value of the following predicate formulas if the domain is the set of all real numbers (the set $\mathbb{R}$):

1. $\forall x \forall y\ P(x,y)$, where $P(x,y)$ is the statement "$x + y = y + x$"
2. $\forall x \exists y\ \ (x + y = 0)$.
3. $\exists y \forall x\ \ (x + y = 0)$.
4. $\forall x \forall y \exists z\ \ (x + y = z)$.
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5. $\exists z \forall x \forall y \; (x + y = z)$.

Solution:

1. If $P(x, y)$ is the statement "$x + y = y + x$", then $\forall x \forall y \; P(x, y)$ means

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Solution:

1. If $P(x,y)$ is the statement "$x + y = y + x$", then $\forall x \forall y \; P(x,y)$ means "$x + y = y + x$ for all real numbers x and y".

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Solution:

1. If $P(x, y)$ is the statement "$x + y = y + x$", then $\forall x \forall y\ P(x, y)$ means "$x + y = y + x$ for all real numbers x and y". According to the commutative law for real numbers addition, we have

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Solution:

1. If $P(x, y)$ is the statement "$x + y = y + x$", then $\forall x \forall y\ P(x, y)$ means "$x + y = y + x$ for all real numbers x and y". According to the commutative law for real numbers addition, we have $\forall x \forall y\ P(x, y) \equiv \mathrm{T}$.

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As a result there is no real number y such that $x + y = 0$ for any value of x.

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As a result there is no real number y such that $x + y = 0$ for any value of x. In other words $\exists y \forall x \ (x + y = 0) \equiv \mathrm{F}$.

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$$z = 3 \text{ and } z = 5, \text{ which is a contradiction.}$$

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   As a result, there is no real number z such that $x + y = z$ for any pair of real numbers x and y.

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As a result, there is no real number z such that $x + y = z$ for any pair of real numbers x and y. In other words $\exists z \forall x \forall y \; (x + y = z) \equiv \mathrm{F}$.

Contents

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Closed Formula

Closed Formula

A predicate formula is a closed formula if all variables occur in that formula are bounded. For example, if P is a binary predicate, x and y are variables, and a and b are concrete elements in the observed domain, then the formulas $\forall x \exists y\ P(x,y)$, $\forall x\ P(x,b)$, and $P(a,b)$ are closed formulas, while $\forall x\ P(x,y)$, $P(x,b)$, and $P(a,y)$ are not closed formulas.

Interpretation

An interpretation for a predicate formula is an assignment of truth for that formula. Unlike propositional formulas, interpretation for predicate formulas depends on the domain or universe of discourse. Interpretations or the truth values of predicate formulas are only defined for closed formula.

Variable Substitution

Variable Substitution

Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A[x \leftarrow d]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula

Variable Substitution

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Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula

Variable Substitution

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Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A[x \leftarrow d]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula "$2(2) \leq 5$"
- $B[y \leftarrow 1]$ is the formula

Variable Substitution

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Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula "$2(2) \leq 5$"
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Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A[x \leftarrow d]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

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Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula "$2(2) \leq 5$"
- $B[y \leftarrow 1]$ is the formula "$(1)^2 \geq 2$", and $B[y \leftarrow 2]$ is the formula "$(2)^2 \geq 2$"
- $A[y \leftarrow 1]$ is the formula

Variable Substitution

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Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A\left[x \leftarrow d\right]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A\left[x \leftarrow 0\right]$ is the formula "$2\left(0\right) \leq 5$", and $A\left[x \leftarrow 2\right]$ is the formula "$2\left(2\right) \leq 5$"
- $B\left[y \leftarrow 1\right]$ is the formula "$\left(1\right)^2 \geq 2$", and $B\left[y \leftarrow 2\right]$ is the formula "$\left(2\right)^2 \geq 2$"
- $A\left[y \leftarrow 1\right]$ is the formula "$2x \leq 5$", and $A\left[y \leftarrow 2\right]$ is the formula

Variable Substitution

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Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A[x \leftarrow d]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

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- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula "$2(2) \leq 5$"
- $B[y \leftarrow 1]$ is the formula "$(1)^2 \geq 2$", and $B[y \leftarrow 2]$ is the formula "$(2)^2 \geq 2$"
- $A[y \leftarrow 1]$ is the formula "$2x \leq 5$", and $A[y \leftarrow 2]$ is the formula "$2x \leq 5$"
- $B[x \leftarrow 0]$ is the formula

Variable Substitution

Variable Substitution

Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A[x \leftarrow d]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A[x \leftarrow 0]$ is the formula "$2(0) \leq 5$", and $A[x \leftarrow 2]$ is the formula "$2(2) \leq 5$"
- $B[y \leftarrow 1]$ is the formula "$(1)^2 \geq 2$", and $B[y \leftarrow 2]$ is the formula "$(2)^2 \geq 2$"
- $A[y \leftarrow 1]$ is the formula "$2x \leq 5$", and $A[y \leftarrow 2]$ is the formula "$2x \leq 5$"
- $B[x \leftarrow 0]$ is the formula "$y^2 \geq 2$", and $B[x \leftarrow 1]$ is the formula

Variable Substitution

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Let A be a predicate formula which is observed in the domain D and let $d \in D$ be a concrete element in D. The notation $A\left[x \leftarrow d\right]$ means a formula which is obtained from replacing **all** occurrence of x by d in formula A.

Example

Suppose A is a formula "$2x \leq 5$" and B is a formula "$y^2 \geq 2$". If the domain is $\{0, 1, 2\}$, then we have

- $A\left[x \leftarrow 0\right]$ is the formula "$2\left(0\right) \leq 5$", and $A\left[x \leftarrow 2\right]$ is the formula "$2\left(2\right) \leq 5$"
- $B\left[y \leftarrow 1\right]$ is the formula "$\left(1\right)^2 \geq 2$", and $B\left[y \leftarrow 2\right]$ is the formula "$\left(2\right)^2 \geq 2$"
- $A\left[y \leftarrow 1\right]$ is the formula "$2x \leq 5$", and $A\left[y \leftarrow 2\right]$ is the formula "$2x \leq 5$"
- $B\left[x \leftarrow 0\right]$ is the formula "$y^2 \geq 2$", and $B\left[x \leftarrow 1\right]$ is the formula "$y^2 \geq 2$"

Interpretation and Its Notation

Suppose D is a domain and A is a predicate formula, the notation $\mathcal{I}_D(A)$ denotes the interpretation of formula A over the domain D. The notation $\mathcal{I}_D(A) = \mathrm{T}$ means formula A is interpreted to true by interpretation $\mathcal{I}$ over the domain D, while $\mathcal{I}_D(A) = \mathrm{F}$ means formula A is interpreted to false by interpretation $\mathcal{I}$ over the domain D.

Semantics Rules of Predicate Formulas

Semantics Rules of Predicate Formulas

Suppose A is a formula, D is the domain (universe of discourse), and $\mathcal{I}$ is an interpretation which is well-defined for every atomic proposition occurring in A. The interpretation of A is defined as follows:

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- If $A = P(d_1, d_2, \ldots, d_n)$ for d_i $(1 \leq i \leq n)$ in the domain, then $\mathcal{I}_D(A) = \mathcal{I}_D(P(d_1, d_2, \ldots, d_n)) = \mathrm{T}$ if there is a relation among $d_1, d_2, \ldots, d_n$ which leads to true according to the definition of predicate P.

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- If $A = \mathrm{T}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{T}\right) = \mathrm{T}$. Analogously, if $A = \mathrm{F}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{F}\right) = \mathrm{F}$.

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- If $A = P\left(d_1, d_2, \ldots, d_n\right)$ for d_i $(1 \leq i \leq n)$ in the domain, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(P\left(d_1, d_2, \ldots, d_n\right)\right) = \mathrm{T}$ if there is a relation among $d_1, d_2, \ldots, d_n$ which leads to true according to the definition of predicate P.
- If $A = \mathrm{T}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{T}\right) = \mathrm{T}$. Analogously, if $A = \mathrm{F}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{F}\right) = \mathrm{F}$.
- If $A = \forall x\ B$ for some formula B, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\forall x\ B\right) = \mathrm{T}$ if $\mathcal{I}_D\left(B\left[x \leftarrow d\right]\right) = \mathrm{T}$ for **all** d in the domain D.

Semantics Rules of Predicate Formulas

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Suppose A is a formula, D is the domain (universe of discourse), and $\mathcal{I}$ is an interpretation which is well-defined for every atomic proposition occurring in A. The interpretation of A is defined as follows:

- If $A = P\left(d_1, d_2, \ldots, d_n\right)$ for d_i $(1 \leq i \leq n)$ in the domain, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(P\left(d_1, d_2, \ldots, d_n\right)\right) = \mathrm{T}$ if there is a relation among $d_1, d_2, \ldots, d_n$ which leads to true according to the definition of predicate P.
- If $A = \mathrm{T}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{T}\right) = \mathrm{T}$. Analogously, if $A = \mathrm{F}$, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\mathrm{F}\right) = \mathrm{F}$.
- If $A = \forall x\ B$ for some formula B, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\forall x\ B\right) = \mathrm{T}$ if $\mathcal{I}_D\left(B\left[x \leftarrow d\right]\right) = \mathrm{T}$ for **all** d in the domain D.
- If $A = \exists x\ B$ for some formula B, then $\mathcal{I}_D\left(A\right) = \mathcal{I}_D\left(\exists x\ B\right) = \mathrm{T}$ if $\mathcal{I}_D\left(B\left[x \leftarrow d\right]\right) = \mathrm{T}$ for **some** d in the domain D.

- If $A = \neg B$, for some formula B, then
$\mathcal{I}(A) = \mathcal{I}(\neg B) = \neg \mathcal{I}(B) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) = \mathrm{F} \\ \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathrm{T} \end{cases}.$

- If $A = \neg B$, for some formula B, then
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- If $A = B \wedge C$, for some formulas B and C, then
$$\mathcal{I}(A) = \mathcal{I}(B \wedge C) = \mathcal{I}(B) \wedge \mathcal{I}(C) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) = \mathcal{I}(C) = \mathrm{T} \\ \mathrm{F}, & \text{otherwise} \end{cases}.$$

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- If $A = B \vee C$, for some formulas B and C, then
$$\mathcal{I}(A) = \mathcal{I}(B \vee C) = \mathcal{I}(B) \vee \mathcal{I}(C) = \begin{cases} \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathcal{I}(C) = \mathrm{F} \\ \mathrm{T}, & \text{otherwise} \end{cases}.$$

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- If $A = B \vee C$, for some formulas B and C, then
$$\mathcal{I}(A) = \mathcal{I}(B \vee C) = \mathcal{I}(B) \vee \mathcal{I}(C) = \begin{cases} \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathcal{I}(C) = \mathrm{F} \\ \mathrm{T}, & \text{otherwise} \end{cases}.$$
- If $A = B \oplus C$, for some formulas B and C, then
$$\mathcal{I}(A) = \mathcal{I}(B \oplus C) = \mathcal{I}(B) \oplus \mathcal{I}(C) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) \neq \mathcal{I}(C) \\ \mathrm{F}, & \text{if } \mathcal{I}(C) = \mathcal{I}(C) \end{cases}.$$

- If $A = \neg B$, for some formula B, then $\mathcal{I}(A) = \mathcal{I}(\neg B) = \neg \mathcal{I}(B) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) = \mathrm{F} \\ \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathrm{T} \end{cases}.$
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- If $A = B \vee C$, for some formulas B and C, then $\mathcal{I}(A) = \mathcal{I}(B \vee C) = \mathcal{I}(B) \vee \mathcal{I}(C) = \begin{cases} \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathcal{I}(C) = \mathrm{F} \\ \mathrm{T}, & \text{otherwise} \end{cases}.$
- If $A = B \oplus C$, for some formulas B and C, then $\mathcal{I}(A) = \mathcal{I}(B \oplus C) = \mathcal{I}(B) \oplus \mathcal{I}(C) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) \neq \mathcal{I}(C) \\ \mathrm{F}, & \text{if } \mathcal{I}(C) = \mathcal{I}(C) \end{cases}.$
- If $A = B \rightarrow C$, for some formulas B and C, then $\mathcal{I}(A) = \mathcal{I}(B \rightarrow C) = \mathcal{I}(B) \rightarrow \mathcal{I}(C) = \begin{cases} \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathrm{T} \text{ but } \mathcal{I}(C) = \mathrm{F} \\ \mathrm{T}, & \text{otherwise} \end{cases}.$

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- If $A = B \vee C$, for some formulas B and C, then
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- If $A = B \oplus C$, for some formulas B and C, then
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- If $A = B \rightarrow C$, for some formulas B and C, then $\mathcal{I}(A) = \mathcal{I}(B \rightarrow C) =$
$$\mathcal{I}(B) \rightarrow \mathcal{I}(C) = \begin{cases} \mathrm{F}, & \text{if } \mathcal{I}(B) = \mathrm{T} \text{ but } \mathcal{I}(C) = \mathrm{F} \\ \mathrm{T}, & \text{otherwise} \end{cases}.$$
- If $A = B \leftrightarrow C$, for some formulas B and C, then
$$\mathcal{I}(A) = \mathcal{I}(B \leftrightarrow C) = \mathcal{I}(B) \leftrightarrow \mathcal{I}(C) = \begin{cases} \mathrm{T}, & \text{if } \mathcal{I}(B) = \mathcal{I}(C) \\ \mathrm{F}, & \text{if } \mathcal{I}(B) \neq \mathcal{I}(C) \end{cases}.$$

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Intuitive Semantics of Predicate Formulas

Exercise

Suppose $P(x)$ "x is odd" and $Q(x)$: "x is even" are two predicate over integers domain, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Determine the truth value for each of these formulas:

1. $\forall x \ (P(x) \vee Q(x))$
2. $\forall x P(x) \vee \forall x Q(x)$
3. $\exists x \ (P(x) \wedge Q(x))$
4. $\exists x P(x) \wedge \exists x Q(x)$

Number 1.
$\forall x \ \ (P(x) \vee Q(x))$ means:

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- Suppose c is an arbitrary integers in $\mathbb{Z}$, then either $P(c) \equiv \mathrm{T}$ or $Q(c) \equiv \mathrm{T}$ but not both.
- Therefore $\forall x\ \ (P(x) \vee Q(x)) \equiv$

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Number 2.
$\forall x P(x) \vee \forall x Q(x)$ means:

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$\forall x P(x) \vee \forall x Q(x)$ means:

- "for every integer x it is $P(x)$ or for every integer x it is $Q(x)$".
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- We have $\forall x P(x) \equiv$

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- Suppose c is an arbitrary integers in $\mathbb{Z}$, then either $P(c) \equiv \mathrm{T}$ or $Q(c) \equiv \mathrm{T}$ but not both.
- Therefore $\forall x\ (P(x) \vee Q(x)) \equiv \mathrm{T}$.

Number 2.
$\forall x P(x) \vee \forall x Q(x)$ means:

- "for every integer x it is $P(x)$ or for every integer x it is $Q(x)$".
- Or "every integer x is odd or every integer x is even".
- We have $\forall x P(x) \equiv \mathrm{F}$ because $P(2) \equiv \mathrm{F}$. We also have $\forall x Q(x) \equiv$

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- Suppose c is an arbitrary integers in $\mathbb{Z}$, then either $P(c) \equiv \mathrm{T}$ or $Q(c) \equiv \mathrm{T}$ but not both.
- Therefore $\forall x \ (P(x) \vee Q(x)) \equiv \mathrm{T}$.

Number 2.
$\forall x P(x) \vee \forall x Q(x)$ means:

- "for every integer x it is $P(x)$ or for every integer x it is $Q(x)$".
- Or "every integer x is odd or every integer x is even".
- We have $\forall x P(x) \equiv \mathrm{F}$ because $P(2) \equiv \mathrm{F}$. We also have $\forall x Q(x) \equiv \mathrm{F}$ because $Q(1) \equiv \mathrm{F}$.
- Therefore $\forall x P(x) \vee \forall x Q(x) \equiv$

Number 1.
$\forall x\ \left(P\left(x\right) \lor Q\left(x\right)\right)$ means:

- "for every x it is $P\left(x\right)$ or $Q\left(x\right)$".
- Or "for every integer x, x is odd or x is even".
- Suppose c is an arbitrary integers in $\mathbb{Z}$, then either $P\left(c\right) \equiv \mathrm{T}$ or $Q\left(c\right) \equiv \mathrm{T}$ but not both.
- Therefore $\forall x\ \left(P\left(x\right) \lor Q\left(x\right)\right) \equiv \mathrm{T}$.

Number 2.
$\forall x P\left(x\right) \lor \forall x Q\left(x\right)$ means:

- "for every integer x it is $P\left(x\right)$ or for every integer x it is $Q\left(x\right)$".
- Or "every integer x is odd or every integer x is even".
- We have $\forall x P\left(x\right) \equiv \mathrm{F}$ because $P\left(2\right) \equiv \mathrm{F}$. We also have $\forall x Q\left(x\right) \equiv \mathrm{F}$ because $Q\left(1\right) \equiv \mathrm{F}$.
- Therefore $\forall x P\left(x\right) \lor \forall x Q\left(x\right) \equiv \mathrm{F}$.

Number 3.
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$\exists x\ \ (P(x) \wedge Q(x))$ means:

- "there is an integer x such that $P(x)$ and $Q(x)$"
- Or "there is an integer x such that x is odd and x is even".
- Suppose c is an integer satisfying this criterion, then c is simultaneously an odd and an even integer.
- Therefore, **there is no** $c \in \mathbb{Z}$ such that $P(c) \wedge Q(c) \equiv \mathrm{T}$, and thus $\exists x\ \ (P(x) \wedge Q(x)) \equiv$

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- Suppose c is an integer satisfying this criterion, then c is simultaneously an odd and an even integer.
- Therefore, **there is no** $c \in \mathbb{Z}$ such that $P(c) \wedge Q(c) \equiv \mathrm{T}$, and thus $\exists x\ (P(x) \wedge Q(x)) \equiv \mathrm{F}$.

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$\exists x P(x) \wedge \exists x Q(x)$ means:

- "there is an integer x such that $P(x)$ and there is an integer x such that $Q(x)$"
- Or "there is an odd integer x and there is an even integer x".
- We have $P(1) \equiv$

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$\exists x \; (P(x) \wedge Q(x))$ means:

- "there is an integer x such that $P(x)$ and $Q(x)$"
- Or "there is an integer x such that x is odd and x is even".
- Suppose c is an integer satisfying this criterion, then c is simultaneously an odd and an even integer.
- Therefore, **there is no** $c \in \mathbb{Z}$ such that $P(c) \wedge Q(c) \equiv \mathrm{T}$, and thus $\exists x \; (P(x) \wedge Q(x)) \equiv \mathrm{F}$.

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- "there is an integer x such that $P(x)$ and there is an integer x such that $Q(x)$"
- Or "there is an odd integer x and there is an even integer x".
- We have $P(1) \equiv \mathrm{T}$, so $\exists x P(x) \equiv$

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- "there is an integer x such that $P(x)$ and there is an integer x such that $Q(x)$"
- Or "there is an odd integer x and there is an even integer x".
- We have $P(1) \equiv \mathrm{T}$, so $\exists x P(x) \equiv \mathrm{T}$. We also have $Q(2) \equiv$

Number 3.
$\exists x \ (P(x) \wedge Q(x))$ means:

- "there is an integer x such that $P(x)$ and $Q(x)$"
- Or "there is an integer x such that x is odd and x is even".
- Suppose c is an integer satisfying this criterion, then c is simultaneously an odd and an even integer.
- Therefore, **there is no** $c \in \mathbb{Z}$ such that $P(c) \wedge Q(c) \equiv \mathrm{T}$, and thus $\exists x \ (P(x) \wedge Q(x)) \equiv \mathrm{F}$.

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- We have $P(1) \equiv \mathrm{T}$, so $\exists x P(x) \equiv \mathrm{T}$. We also have $Q(2) \equiv \mathrm{T}$, so $\exists x Q(x) \equiv \mathrm{T}$.
- Therefore $\exists x P(x) \wedge \exists x Q(x) \equiv \mathrm{T} \wedge \mathrm{T} \equiv \mathrm{T}$.

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Validity, Satisfiability, and Contradiction

Definition

Let A be a predicate formula

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1. A is *valid* iff A is **true** (T) for **any interpretation** defined over **any domain**. In this case, A is also called a tautology.

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3. A is *contradictory/ unsatisfiable* iff A is **false** (F) for **any interpretation** defined over **any domain**. In this case, A is also called a contradiction.
4. A is a *contingency* iff A is neither valid nor contradictory.

Unlike the validity, the satisfiability, and the contradictory in propositional logic, proving the validity, the satisfiability, and the contradictory in predicate logic cannot always be performed using truth table method.

Proving Validity in Predicate Logic

Example

If P and Q are unary predicates, then $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x (P(x) \wedge Q(x))$ is valid.

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5. Therefore $\mathcal{I}_D(\forall x (P(x) \wedge Q(x))) = \mathrm{T}$.

Exercise

Prove that $\forall x\ (P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)$ is valid.

Proving Contradictory in Predicate Logic

Example

If P is a unary predicate, then $\forall x P(x) \rightarrow \exists x \neg P(x)$ is a contradiction.

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If P is a unary predicate, then $\forall x P(x) \rightarrow \exists x \neg P(x)$ is a contradiction. We will prove that the truth of $\forall x P(x)$ never leads to the truth of $\exists x \neg P(x)$.

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4. From no. 2 we also have $\mathcal{I}_D(P(c)) = \mathrm{T}$ (because d in no. 2 is arbitrary, we may choose $d = c$).
5. The results in no. 3 and 4 are inconsistent, therefore there is no interpretation $\mathcal{I}$ and domain D such that $\mathcal{I}_D(\forall x P(x) \rightarrow \exists x \neg P(x)) = \mathrm{T}$.

Proving Contradictory in Predicate Logic

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6. No. 5 means $\forall x P(x) \rightarrow \exists x \neg P(x)$ is a contradiction.

Exercise

Prove that $\exists x \neg P(x) \rightarrow \forall x P(x)$ is a contradiction

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Logical Consequence and Logical Equivalence

Definition

Suppose A and B are two predicate formulas.
Formula A and B are *(logically) equivalent* if the formula

$$A \leftrightarrow B$$

is a **tautology**. In this condition, we write $A \equiv B$ or $A \Leftrightarrow B$.
Formula B is said to be the *(logical) consequence* of A if the formula

$$A \rightarrow B$$

is a **tautology**. In this condition, we write $A \Rightarrow B$.

Unlike in propositional logic, we cannot use truth table for proving the logical consequence or logical equivalence between two predicate formulas.

Examples of Logical Consequence and Logical Equivalence

Example

Let P and Q be two unary predicates. Earlier in this slide, we've proved that $\forall x P(x) \wedge \forall x Q(x) \rightarrow \forall x (P(x) \wedge Q(x))$ is a tautology, therefore we have $\forall x P(x) \wedge \forall x Q(x) \Rightarrow \forall x (P(x) \wedge Q(x))$

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In addition, we can also prove that $\forall x (P(x) \wedge Q(x)) \rightarrow \forall x P(x) \wedge \forall x Q(x)$ is also tautology (left as an exercise for the reader), then we have $\forall x (P(x) \wedge Q(x)) \Rightarrow \forall x P(x) \wedge \forall x Q(x)$.

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From these results we obtain $\forall x P(x) \wedge \forall x Q(x) \Leftrightarrow \forall x \ (P(x) \wedge Q(x))$, or in another notation $\forall x P(x) \wedge \forall x Q(x) \equiv \forall x (P(x) \wedge Q(x))$.

Exercise

Prove that if P and Q are unary predicates, then
$\exists x P(x) \vee \exists x Q(x) \equiv \exists x (P(x) \vee Q(x))$.

Solution:

Exercise

Prove that if P and Q are unary predicates, then $\exists x P(x) \vee \exists x Q(x) \equiv \exists x \left(P(x) \vee Q(x)\right)$.

Solution: We shall prove that $\exists x P(x) \vee \exists x Q(x) \Rightarrow \exists x \ \left(P(x) \vee Q(x)\right)$ and $\exists x \left(P(x) \vee Q(x)\right) \Rightarrow \exists x P(x) \vee \exists x Q(x)$.

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Prove that if P and Q are unary predicates, then $\exists x P(x) \vee \exists x Q(x) \equiv \exists x (P(x) \vee Q(x))$.

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We shall show that $\exists x (P(x) \vee Q(x)) \Rightarrow \exists x P(x) \vee \exists x Q(x)$, or in other words $\exists x (P(x) \vee Q(x)) \rightarrow \exists x P(x) \vee \exists x Q(x)$ is valid,

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5. Therefore, $\mathcal{I}_D\left(\exists x\left(P\left(x\right) \vee Q\left(x\right)\right)\right) = \mathrm{T}$ implies $\mathcal{I}_D\left(\exists x P\left(x\right) \vee \exists x Q\left(x\right)\right) = \mathrm{T}$, hence $\exists x\left(P\left(x\right) \vee Q\left(x\right)\right) \rightarrow \exists x P\left(x\right) \vee \exists x Q\left(x\right)$ is valid, or in other words $\exists x\left(P\left(x\right) \vee Q\left(x\right)\right) \Rightarrow \exists x P\left(x\right) \vee \exists x Q\left(x\right)$.

Contents

1 Truth of Formulas with Single Quantifier

2 Truth of Formulas with Two/ More Quantifiers

3 Interpretation and Semantics of Predicate Formulas (Supplementary)

4 Intuitive Semantics of Predicate Formulas

5 Predicate Formulas Based on Their Semantics (Supplementary)

6 Logical Consequence and Logical Equivalence (Supplementary)

7 Equivalences for Predicate Formulas: Negation of Quantified Formulas

Logical Equivalences in Predicate Logic

Predicate logic can be considered as an "extension" of propositional logic, as a result all logical equivalences in propositional logic are also applied for predicate formulas.

For example, since in propositional logic we have $\neg (A \wedge B) \equiv \neg A \vee \neg B$ and $A \rightarrow B \equiv \neg A \vee B$ for any propositional formulas A and B, then in predicate logic these logical equivalences are also correct. For instance, if P and Q are unary predicates, then

$$\exists x \left(\neg \left(P\left(x\right) \wedge Q\left(x\right)\right)\right) \quad \equiv$$

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$$\begin{aligned} \exists x \left(\neg \left(P(x) \wedge Q(x)\right)\right) &\equiv \exists x \left(\neg P(x) \vee \neg Q(x)\right) \\ \forall x \left(P(x) \rightarrow Q(x)\right) &\equiv \forall x \left(\neg P(x) \vee Q(x)\right) \end{aligned}$$

In addition to all propositional equivalences, predicate logic has two additional equivalences concerning the negation of quantified formulas.

Negation of Universal Quantification

Suppose we want to determine the negation of following sentence: "every informatics student takes Mathematical Logic class".

The above sentence can be translated into predicate formula as $\forall x\ P(x)$, with the domain D for x is the set of all students in informatics major and P is a unary predicate "takes Mathematical Logic class".

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- The negation of $\forall x\ P(x)$ is a formula which is **true** precisely when $\forall x\ P(x)$ is **false**. Recall that if $\forall x\ P(x)$ is **false**, then there is at least one $x \in D$ such that $P(x)$ is **false**.

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- Since $\forall x\ P(x)$ is false precisely when $\exists x\ \neg P(x)$ is true, then we have $\neg \forall x\ P(x) \equiv$

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- Since $\forall x\ P(x)$ is false precisely when $\exists x\ \neg P(x)$ is true, then we have $\boxed{\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)}$.

Therefore, the negation of the above sentence is "there is an informatics student who doesn't take Mathematical Logic class".

Negation of Existential Quantification

Now, suppose we want to determine the negation of following sentence: "there is an informatics student who takes Formal Methods class".

The above sentence can be translated into predicate formula as $\exists x\ P(x)$, with the domain D for x is the set of all students in informatics major and P is a unary predicate "takes Formal Methods class".

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Therefore, the negation of the above sentence is "every informatics student doesn't take Formal Methods class".

De Morgan's Laws for Quantifier

Suppose P is a unary predicate defined over a finite domain $D = \{a_1, a_2, \ldots, a_n\}$. We have

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Analogously, we can obtain $\neg\exists x \; P(x) \equiv \forall x \neg P(x)$. The equivalences $\neg\forall x \; P(x) \equiv \exists x \; \neg P(x)$ and $\neg\exists x \; P(x) \equiv \forall x \; \neg P(x)$ are called De Morgan's laws for quantifiers.

Exercise: Negation of Quantified Formulas

Exercise

Express the negation of each of these predicate formulas so that no negation precedes a quantifier.

1. $\forall x \ \left(x^2 > 0\right)$
2. $\exists y \ (y + 1 \neq 2)$
3. $\forall x \exists y \ (xy = 1)$
4. $\exists x \forall y \ (x + y \neq 1)$
5. $\forall x \forall y \ \left((xy)^2 \leq 0\right)$

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Exercise: Negation of Quantified Formulas

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Express the negation of each of these predicate formulas so that no negation precedes a quantifier.

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