Basic Counting Techniques

Discrete Mathematics – Second Term 2022-2023

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications* , 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

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Motivation: What do we count?

Notice the following problems.

"Counting" Problem

1. Password that we need in an online forum must contain 6, 7, or 8 characters. Each character is a digit of decimal number or capital letter within the alphabet A-Z. Each password must contain at least one decimal number.

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1. Password that we need in an online forum must contain 6, 7, or 8 characters. Each character is a digit of decimal number or capital letter within the alphabet A-Z. Each password must contain at least one decimal number. How many different passwords are there?
2. An Indonesian national football team consists of 23 players. Three of them are goalkeepers.

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2. An Indonesian national football team consists of 23 players. Three of them are goalkeepers. How many different starting lineups that we can have if exactly one goalkeeper must play?
3. PIN of ATM of a bank consists of 6 digits $(0-9)$. If the bank has 50 million customers, how many

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3. PIN of ATM of a bank consists of 6 digits $(0-9)$. If the bank has 50 million customers, how many people must be gathered to ensure that at least two customers have identical PINs?

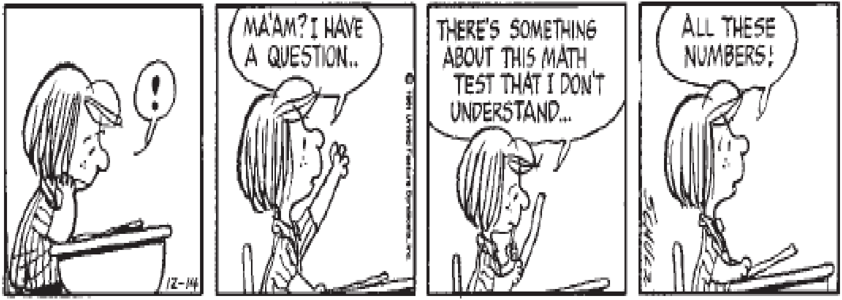
Combinatorics: branch of math dealing with combinations of objects, the most important part in Discrete Math.
Enumeration: counting an object with particular properties, the most important part in combinatorics.

Counting

Counting . . . is not as easy as it sounds.

Counting

Counting ... is not as easy as it sounds.



Counting is not as easy as it sounds, but when one knows exactly what to count, the counting itself is as easy as $1-2-3$.

Contents

Sum Rule

Sum Rule/ Addition Rule

Suppose there are two tasks that must be done, namely T_1 and T_2. Task T_1can be done in n_1 ways, task T_2 can be done in n_2 ways, and the two tasks cannot be done simultaneously, then there are

$$n_1 + n_2$$

ways to complete the tasks.
Suppose there are m tasks that must be completed, namely $T_1, T_2, \ldots, T_m$. Each task T_i can be done in n_i ways and there are no two different tasks that can be done simultaneously, then there are

$$n_1 + n_2 + \cdots + n_m = \sum_{i=1}^{m} n_i$$

ways to do the tasks.

Set Representation of Sum Rule

Sum Rule/ Addition Rule

Given some finite disjoint sets $A_1, A_2, \ldots, A_m$ (i.e., $A_i \cap A_j = \emptyset$ for **every** $i, j \in \{1, 2, \ldots, m\}$ where $i \neq j$), then the number of ways to choose **one** member of $A_1 \cup A_2 \cup \cdots \cup A_m$ is the sum of the cardinality of each sets.

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_m| &= |A_1| + |A_2| + \cdots + |A_m| \\ \left|\bigcup_{i=1}^{m} A_i\right| &= \sum_{i=1}^{m} |A_i| . \end{aligned}$$

Problem Example: Sum Rule

Problem example: sum rule

1. In a class there are 25 male students and 15 female students. Determine how many ways to choose a class representative.
2. A company that sells laptop wants to give a laptop to a student in a class. If there are 20 electrical engineering students, 30 informatics students, and 10 industrial engineering students, in how many different ways the laptop can be given?
3. A restaurant sells various cuisines as follows: 10 Indonesian cuisines, 10 Middle Eastern cuisines, 5 Oriental cuisines, and 3 European cuisines. Suppose you have a voucher of free meal that can be used for one lunch. How many different menus you can choose?

Solutions

Solutions:

1. There are $25 + 15 =$

Solutions

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Solutions

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1. There are $25 + 15 = 40$ students in the class, therefore there are 40 different ways to choose a representative.
2. Notice that: there are 20 ways to give the laptop to electrical engineering students, there are 30 ways to give the laptop to informatics students, and there are 10 ways to give the laptop to industrial engineering students. Because there is no student enrolled in two different programs, then there are

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1. There are $25 + 15 = 40$ students in the class, therefore there are 40 different ways to choose a representative.
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3. Notice that: there are 10 ways to choose Indonesian cuisine, there are 10 ways to choose Middle Eastern cuisine, there are 5 ways to choose Oriental cuisine, and there are

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3. Notice that: there are 10 ways to choose Indonesian cuisine, there are 10 ways to choose Middle Eastern cuisine, there are 5 ways to choose Oriental cuisine, and there are 3 ways to choose European cuisine. Assuming that there is no food registered in two different menus, there are $10 + 10 + 5 + 3 = 28$ ways to choose a lunch meal.

Contents

Product Rule

Product Rule/ Multiplication Rule

Suppose a procedure can be divided into two consecutive tasks, namely T_1 and T_2. If there are n_1 ways to do T_1 and n_2 ways to do T_2, then there are

$$n_1 \cdot n_2$$

ways to do the procedure.

Suppose a procedure can be divided into a sequence of tasks $T_1, T_2, \ldots, T_m$ where each job can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are

$$n_1 \cdot n_2 \cdots n_m = \prod_{i=1}^{m} n_i$$

ways to do the procedure.

Set Representation of Product Rule

Product Rule/ Multiplication Rule

Given some finite sets $A_1, A_2, \ldots, A_m$, then the number of ways to choose **one** member of Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is by choosing one member of A_1, one member of A_2, ..., **and** one member of A_m.

$$\begin{aligned} |A_1 \times A_2 \times \cdots \times A_m| &= |A_1| \cdot |A_2| \cdots |A_m| \\ &= \prod_{i=1}^{m} |A_i| . \end{aligned}$$

Problem Example: Product Rule

Problem example: product rule

1. In a class there are 25 male students and 15 female students. Determine the number of ways to choose a male representative and his female vice representative.
2. A cafeteria offers breakfast, lunch, and dinner menu as follows:
   1. breakfast menu: chicken porridge, fried rice, toast
   2. lunch menu: burger, fried rice, curry rice, spaghetti
   3. dinner menu: roasted fish, fried rice, pizza, spaghetti.

   How many menu combinations of breakfast, lunch, and dinner are there?
3. A license plate in a country is started with a capital letter (from A-Z), followed by 3 digits decimal numbers (0-9), and ended by two capital letters (from A-Z). How many different license plates in that country?

Solutions

Solutions:

1. Let $M = \{x : x \text{ a male student in the class}\}$ and $F = \{x : x \text{ a female student in the class}\}$.

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1. Let $M = \{x : x \text{ a male student in the class}\}$ and $F = \{x : x \text{ a female student in the class}\}$. A representative and a his vice is a member of $M \times F$. The number of possibilities are $|M \times F| = |M| \cdot |F| = 25 \cdot 15 = 375$ pairs of representative and his vice.

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2. Let $B = \{\text{chicken porridge, fried rice, toast}\}$, $L = \{\text{burger, fried rice, curry rice, spaghetti}\}$, $D = \{\text{roasted fish, fried rice, pizza, spaghetti}\}$.

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2. Let $B = \{\text{chicken porridge, fried rice, toast}\}$, $L = \{\text{burger, fried rice, curry rice, spaghetti}\}$, $D = \{\text{roasted fish, fried rice, pizza, spaghetti}\}$. A Cartesian product $B \times L \times D$ denotes a combination of breakfast, lunch, and dinner. As an example, some of these combinations of breakfast, lunch, and dinner are

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2. Let $B = \{\text{chicken porridge, fried rice, toast}\}$, $L = \{\text{burger, fried rice, curry rice, spaghetti}\}$, $D = \{\text{roasted fish, fried rice, pizza, spaghetti}\}$. A Cartesian product $B \times L \times D$ denotes a combination of breakfast, lunch, and dinner. As an example, some of these combinations of breakfast, lunch, and dinner are (chicken porridge, burger, roasted fish), (fried rice, fried rice, fried rice), and (toast, curry rice, pizza). Based on the product rule, the number of combination of breakfast, lunch, and dinner are $|B \times L \times D| = |B| \cdot |L| \cdot |D| = 3 \cdot 4 \cdot 4 = 48$ combinations.

3. Notice that the license plate in the country is of the form

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p_1	p_2	p_3	p_4	p_5	p_6

, where

    1. p_1 is a capital letter (from A-Z), therefore, there are

3. Notice that the license plate in the country is of the form

p_1	p_2	p_3	p_4	p_5	p_6

, where

   1. p_1 is a capital letter (from A-Z), therefore, there are 26 ways to choose p_1
   2. p_2, p_3, p_4 are digits of decimal numbers, therefore, there are

3. Notice that the license plate in the country is of the form

p_1	p_2	p_3	p_4	p_5	p_6

, where

   1. p_1 is a capital letter (from A-Z), therefore, there are 26 ways to choose p_1
   2. p_2, p_3, p_4 are digits of decimal numbers, therefore, there are 10 ways to choose for each p_2, p_3, p_4
   3. p_5 and p_6 are capital letters (from A-Z), therefore, there are

3. Notice that the license plate in the country is of the form

p_1	p_2	p_3	p_4	p_5	p_6

, where

   1. p_1 is a capital letter (from A-Z), therefore, there are 26 ways to choose p_1
   2. p_2, p_3, p_4 are digits of decimal numbers, therefore, there are 10 ways to choose for each p_2, p_3, p_4
   3. p_5 and p_6 are capital letters (from A-Z), therefore, there are 26 ways to choose for each p_5 and p_6.

Based on the product rule, the number of different license plates are

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   2. p_2, p_3, p_4 are digits of decimal numbers, therefore, there are 10 ways to choose for each p_2, p_3, p_4
   3. p_5 and p_6 are capital letters (from A-Z), therefore, there are 26 ways to choose for each p_5 and p_6.

Based on the product rule, the number of different license plates are $26 \cdot 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 = (260)^3 = 17\,576\,000$ plates.

Contents

Problem 1

Exercise

1. Given non empty finite sets A and B. If $|A| = m$ and $|B| = n$, determine the number of different total functions from A to B. Furthermore, determine the number of different total functions from B to A.
2. In a school there are m girls and n boys with $m < n$. The school will have a dancing party. Each girl chooses exactly one boy to accompany her to the dancing party. Determine the number of possible combinations of dancing pairs.
3. A 1980s computer can be activated using a password that consists of 6 characters. Each character is a capital letter (A-Z) or a digit of decimal number. If a password must contain at least one digit of decimal number, how many *possible password are there*?

Solution of Problem 1 No. 1

Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

Solution of Problem 1 No. 1

Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B.

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Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is

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Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \rightarrow B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
- the number of choices for $f(a_2)$ is

Solution of Problem 1 No. 1

Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
- the number of choices for $f(a_2)$ is n,
- $\vdots$
- and the number of choices for $f(a_m)$ is

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Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
- the number of choices for $f(a_2)$ is n,
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Therefore, based on the product rule, there are

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Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
- the number of choices for $f(a_2)$ is n,
- $\vdots$
- and the number of choices for $f(a_m)$ is n.

Therefore, based on the product rule, there are $\prod_{i=1}^{m} n = n^m$ different total functions from A to B.

Solution of Problem 1 No. 1

Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
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- and the number of choices for $f(a_m)$ is n.

Therefore, based on the product rule, there are $\prod_{i=1}^{m} n = n^m$ different total functions from A to B. Using a similar reasoning, the number of different total functions from B to A is

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Since $|A| = m$ and $|B| = n$, without loss of generality, we assume $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Firstly, we count the number of different total functions from A to B. Let $f : A \to B$ is a total function, notice that:

- the number of choices for $f(a_1)$ is n,
- the number of choices for $f(a_2)$ is n,
- $\vdots$
- and the number of choices for $f(a_m)$ is n.

Therefore, based on the product rule, there are $\prod_{i=1}^{m} n = n^m$ different total functions from A to B. Using a similar reasoning, the number of different total functions from B to A is m^n.

Solution of Problem 1 No. 2

Let

$$\begin{aligned} X &= \{x : x \text{ a girl in the school}\}, \\ Y &= \{y : y \text{ a boy in the school}\}. \end{aligned}$$

Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose

Solution of Problem 1 No. 2

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Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose n people from $y_1, y_2 \ldots, y_n$.

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Let

$$X = \{x : x \text{ a girl in the school}\},$$
$$Y = \{y : y \text{ a boy in the school}\}.$$

Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose n people from $y_1, y_2 \ldots, y_n$.
- Since x_2 cannot choose the one that has been chosen by x_1, then there are

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Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose n people from $y_1, y_2 \ldots, y_n$.
- Since x_2 cannot choose the one that has been chosen by x_1, then there are $n - 1$ choices for x_2.

Solution of Problem 1 No. 2

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$$X = \{x : x \text{ a girl in the school}\},$$
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Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose n people from $y_1, y_2 \ldots, y_n$.
- Since x_2 cannot choose the one that has been chosen by x_1, then there are $n-1$ choices for x_2.
- Since x_3 cannot choose the one that has been chosen by x_1 nor x_2, then there are

Solution of Problem 1 No. 2

Let

$$X = \{x : x \text{ a girl in the school}\},$$
$$Y = \{y : y \text{ a boy in the school}\}.$$

Since $|X| = m$ and $|Y| = n$, without loss of generality, we assume $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Notice that

- x_1 can choose n people from $y_1, y_2 \ldots, y_n$.
- Since x_2 cannot choose the one that has been chosen by x_1, then there are $n - 1$ choices for x_2.
- Since x_3 cannot choose the one that has been chosen by x_1 nor x_2, then there are $n - 2$ choices for x_3.

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- And so forth, such that for every $i = 2, \ldots, m$, x_i cannot choose the one that has been chosen by $x_1, \ldots, x_{i-1}$. Therefore, in general, the number of dancing pair choices for x_i where $i = 1, 2, \ldots, m$ is

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Based on the product rule, the number of possible dancing pairs is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-(m-1)) =$$

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The last expression is known as m-permutation of n different objects. Notice that

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$$\begin{aligned} n \cdot (n-1) \cdots (n-m+1) &= \frac{n \cdot (n-1) \cdots (n-m+1)\,(n-m)!}{(n-m)!} \\ &= \frac{n!}{(n-m)!} \end{aligned}$$

The last form is also known with the notation $P(n,m)$ or P_m^n or ${}_nP_m$ or ${}^nP_m$ or $P_{n,m}$.

(Note: basically, the problem is equal with the problem to count the number of total functions that have injective properties from X to Y where $|X| < |Y|$, see the textbook).

Solution of Problem 1 No. 3

Let P : $\#$ passwords of length 6 that contain at least one digit of decimal number. We can find P directly, but it is lengthy and tedious. We can find P using the following way:

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$$P = S - Q,$$

1. S : # of possible string combinations of length 6 over 36 characters (26 alphabet and 10 decimal numbers).
2. Q : # of string combinations with the of length 6 that has no decimal number.

Based on the product rule, we have $S = 36^6$ and $Q = 26^6$. So $P = 36^6 - 26^6 = 1\,867\,866\,560$.

Problem 2

Exercise

1. A binary string or bit string is a string that contains only characters over the set $\{0, 1\}$. The length of a string is the number of digits on the string. For example, string 10110 is a binary string of length 5. Determine the number of binary strings of length 8.
2. A 1980s computer can be activated using a password that consists of 6, 7, or 8 characters. Each character is a capital letter (A-Z) or a digit of decimal number. How many possible passwords are there?
3. A password that we need in a system must contain 6, 7, or 8 characters. Each character is a digit of decimal number or a capital letter within the alphabet A-Z. Each password must contain at least one decimal number. How many different passwords are there?

Solution of Problem 2 No. 1

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every s_i for $1 \leq i \leq 8$ has two possibilities, namely 0 or 1. Based on the product rule, there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8 = 256$$

bit strings of length 8.

Solution of Problem 2 No. 2

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Therefore $P = P_6 + P_7 + P_8$.

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Hence

$$\begin{aligned} P &= P_6 + P_7 + P_8 \\ &= 36^6 + 36^7 + 36^8 \\ &= 36^6 \left(1 + 36 + 36^2\right) \\ &= 2\,901\,650\,853\,888. \end{aligned}$$

Solution of Problem 3 No. 2

Let P : $\#$ of passwords of length 6, 7, or 8 that contain at least one digit of decimal number.

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$$P_i = S_i - Q_i$$

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Therefore

$$\begin{aligned} P &= P_6 + P_7 + P_8 \\ &= (S_6 - Q_6) + (S_7 - Q_7) + (S_8 - Q_8) \\ &= \left(36^6 - 26^6\right) + \left(36^7 - 26^7\right) + \left(36^8 - 26^8\right) \\ &= 2\,684\,483\,063\,360. \end{aligned}$$

Challenging Problems

Challenging Problems

1. A hacker wants to know an administrator password of a forum. The hacker will apply a *brute-force* algorithm (an exhaustive search that tries every possible password). He knows that:
   1. the password contains 8 to 12 characters, each character is a number, a capital letter, or a lowercase letter,
   2. the password **should not contain all numbers or all letters**.

   If the hacker algorithm can try 100 passwords in 1 second, determine the maximum duration that he needs to find the right password.
2. In a class of a school there are 10 girls and 15 boys. The school will have a dance party. Each girl will choose at most one boy in her class to accompany her to the dance party (she may choose a boy from the other class). Determine how many possible pairs of dancing in the class.

Contents

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Problem

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Notice that if s is a binary string of length 8 that satisfies the criterion, then s is of the form

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Task 1: binary string that start with 1

The construction of binary string that start with 1 of length 8 (i.e., $1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$) can be done as follows:

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Based on the product rule, task 1 can be done in $2^7 = 128$ ways

Task 2: binary string that end with 00

The construction of binary string that end with 00 of length 8 (i.e., $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ 0\ 0$) can be done as follows:

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- there are two ways to choose the i-th digit for $i = 1, \ldots, 6$,
- there is one way each for choosing the 7th and 8th digits and both of them must be 0).

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- there is one way each for choosing the 7th and 8th digits and both of them must be 0).

Based on the product rule, task 2 can be finished in $2^6 = 64$ ways.

Can we conclude that (by sum rule) there are $128 + 64 = 192$ binary strings of length 8 that start with 1 or end with 00?

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Task 1 and 2 simultaneously: binary string that start with 1 and end with 00

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Can we conclude that (by sum rule) there are $128 + 64 = 192$ binary strings of length 8 that start with 1 or end with 00? No, because here task 1 and task 2 can be done simultaneously. Therefore, the regular sum rule cannot be applied here. In order to use the sum rule, we need to subtract the result we obtain with the number of cases when task 1 and tasks 2 are done at the same time.

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- there are two ways for choosing the i-th digit for $i = 2, 3, \ldots, 6$.

Based on the product rule, task 1 and 2 can be performed simultaneously in $2^5 = 32$ cases.

We can conclude as follows:
Since there are 128 ways to do task 1 and 64 ways to do task 2, and there are 32 cases when the task 1 and task 2 are completed simultaneously, then there are

We can conclude as follows:
Since there are 128 ways to do task 1 and 64 ways to do task 2, and there are 32 cases when the task 1 and task 2 are completed simultaneously, then there are

$$128 + 64 - 32 = 160$$

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We can conclude as follows:
Since there are 128 ways to do task 1 and 64 ways to do task 2, and there are 32 cases when the task 1 and task 2 are completed simultaneously, then there are

$$128 + 64 - 32 = 160$$

ways to do the task 1 or task 2. In other words, there are 160 binary strings of length 8 start with 1 or end with 00.
In set theory, the above rule is related to the following theorem.

Theorem (Substraction rule)

If A and B are two non-disjoint finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Subtraction Rule/ Inclusion-Exclusion Principle

We have

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - (|A \cap B| + |A \cap C| + |B \cap C|) \\ &\quad + (|A \cap B \cap C|) . \end{aligned}$$

Using mathematical induction we have the following theorem.

Theorem (Inclusion-Exclusion Principle)

If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| =$$

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If $A_1, A_2, \ldots, A_n$ are finite sets, then

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Problem 3

Exercise

How many positive integers no greater than 100 that are divisible by 6 or 9?

Solution: Notice that there are 100 positive integers no greater than 100.

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We have $|A| = \left\lfloor \frac{100}{6} \right\rfloor = 16$ and $|B| = \left\lfloor \frac{100}{9} \right\rfloor = 11$. Notice that

$$\begin{aligned} A \cap B &= \{x \in \mathbb{N} : x \leq 100 \text{ and } x \text{ is divisible by } 6 \text{ and } 9\} \\ &= \end{aligned}$$

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Thus, $|A \cap B| = \left\lfloor \frac{100}{18} \right\rfloor = 5$. Based on inclusion-exclusion principle we have

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How many positive integers no greater than 100 that are divisible by 6 or 9?

Solution: Notice that there are 100 positive integers no greater than 100. Suppose

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Thus, $|A \cap B| = \left\lfloor \frac{100}{18} \right\rfloor = 5$. Based on inclusion-exclusion principle we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 16 + 11 - 5 = 22.$$

Problem 3

Exercise

How many positive integers no greater than 100 that are divisible by 6 or 9?

Solution: Notice that there are 100 positive integers no greater than 100. Suppose

$$\begin{aligned} A &= \{x \in \mathbb{N} : x \leq 100 \text{ and } x \text{ is divisible by } 6\}, \\ B &= \{x \in \mathbb{N} : x \leq 100 \text{ and } x \text{ is divisible by } 9\}. \end{aligned}$$

We have $|A| = \left\lfloor \frac{100}{6} \right\rfloor = 16$ and $|B| = \left\lfloor \frac{100}{9} \right\rfloor = 11$. Notice that

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Thus, $|A \cap B| = \left\lfloor \frac{100}{18} \right\rfloor = 5$. Based on inclusion-exclusion principle we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 16 + 11 - 5 = 22.$$

Hence, there are 22 positive integers no greater than 100 that are divisible by 6 or 9.

Challenging Problems

Challenging Problems

1. Determine how many bit strings of length 8 that satisfies the following criteria:
    1. the bit string start with three digits of 0 or end with two digits of 1 (example: 00010100, 10110111, as well as 00010111);
    2. the bit string start with three digits of 0 or end with two digits of 1, but not both (example: 00010100 as well as 10110111, however, 00010111 is not under this criterion because 00010111 start with three digits of 0 **and** end with two digits of 1).
2. Determine how many positive integers between 100 and 200 (inclusive) that are divisible by one of the following numbers: $2, 3, 5$.

Contents

Division Rule

We have seen some counting rules that involve addition, multiplication, and subtraction operation. Now we will see the counting rule that involve division operation.

Division Rule

There are n/d ways to do the task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to the way w.

In other words, suppose a task can be done in n ways; if in fact for each way there are d ways with identical results and d divides n, then the task can be completed in $\frac{n}{d}$ different ways.

Example

Suppose there are 4 chairs around a round table. Determine how many different ways of sitting for 4 people if two ways of sitting are regarded as identical as long the left neighbor and the right neighbor of each person are not changed.

Solution of Example: Cyclic Permutation (1)

Solution:
Firstly, we label one of the chairs with 1. Then we label the rest of them with 2, 3, and 4 in a clockwise direction. Suppose the people that will sit are k_1, k_2, k_3, k_4. Notice that

1. There are 4 choices for k_1,

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1. There are 4 choices for k_1,
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Using product rule, we have $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways of sitting.

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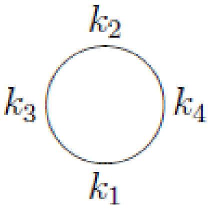
Using product rule, we have $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways of sitting. Basically, this 24 ways of sitting represents the structure of the number of ways of sitting for k_1, k_2, k_3, and k_4 in a linear configuration (in a row, not cyclic).

Solution of example: Cyclic permutation (2)

We suppose the configuration of chairs that we have is as follows.



Notice that all the following configurations are similar to the configuration as in the above picture:

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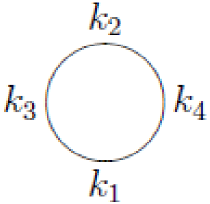

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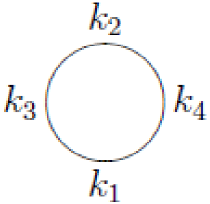

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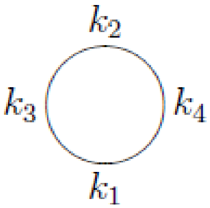

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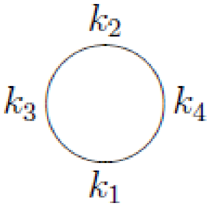

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Generally, if $a, b, c, d \in \{1, 2, 3, 4\}$ and $a \neq b \neq c \neq d$, then all the following configurations give a similar configuration of chairs (right and left neighbor are not changed)

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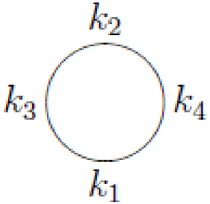

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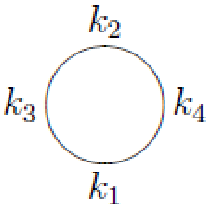

Notice that all the following configurations are similar to the configuration as in the above picture:

$$(k_2, k_4, k_1, k_3), \ (k_3, k_2, k_4, k_1), \ (k_1, k_3, k_2, k_4), \ (k_4, k_1, k_3, k_2).$$

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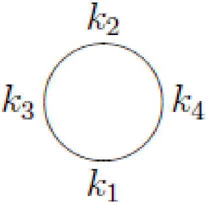

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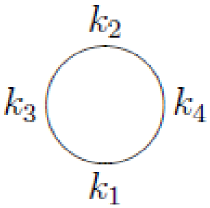

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Generally, if $a, b, c, d \in \{1, 2, 3, 4\}$ and $a \neq b \neq c \neq d$, then all the following configurations give a similar configuration of chairs (right and left neighbor are not changed)

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Solution of example: Cyclic permutation (3)

Therefore, based on the division rule, there are

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Therefore, based on the division rule, there are $\frac{24}{4} = 6$ different ways of sitting for 4 people around a round table.

Theorem (Cyclic Permutation)

Suppose there are n chairs around a round table. Two ways of sitting are regarded as identical as long the left and right neighbor of each person are not changed.

Therefore, the number of different ways of sitting for n people is

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Theorem (Cyclic Permutation)

Suppose there are n chairs around a round table. Two ways of sitting are regarded as identical as long the left and right neighbor of each person are not changed. Therefore, the number of different ways of sitting for n people is $\frac{n!}{n} = (n-1)!$.

Proof

Do it yourself!

Contents

Problem 4

Exercise (Use the previously discussed counting technique.)

1. Determine how many strings with four characters that we can have from the word **BOOK** if: each letter in the word **BOOK** can only be used once.
   (Example: BOOK, BKOO, KOOB, etc.).
2. Determine how many different strings that we can have from the word **BOOKKEEPER** if all letters in the word **BOOKKEEPER** must be used and can only be used once.
   (Example: BOOKKEEPER, KEEPERBOOK, PEEKERKBOO, etc.).

Solutions: Problem 4 No.1

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To make it easier, we first find the number of strings of four characters that we can have from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ (we differentiate the two letters of **O**). Notice that the four characters string is of the form $s_1\ s_2\ s_3\ s_4$ with $s_1, s_2, s_3, s_4 \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}\}$.

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1. the number of possibility of letter at s_1 :

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1. the number of possibility of letter at s_1 : 4,
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1. the number of possibility of letter at $s_1 : 4$,
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3. the number of possibility of letter at $s_3 : 2$,
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2. the number of possibility of letter at $s_2 : 3$,
3. the number of possibility of letter at $s_3 : 2$,
4. the number of possibility of letter at $s_4 : 1$.

Based on the product rule, the number of possible strings from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ is

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To make it easier, we first find the number of strings of four characters that we can have from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ (we differentiate the two letters of **O**). Notice that the four characters string is of the form $s_1\ s_2\ s_3\ s_4$ with $s_1, s_2, s_3, s_4 \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}\}$. Based on the requirement, we have:

1. the number of possibility of letter at $s_1 : 4$,
2. the number of possibility of letter at $s_2 : 3$,
3. the number of possibility of letter at $s_3 : 2$,
4. the number of possibility of letter at $s_4 : 1$.

Based on the product rule, the number of possible strings from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ is $4!$.

Solutions: Problem 4 No.1

To make it easier, we first find the number of strings of four characters that we can have from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ (we differentiate the two letters of **O**). Notice that the four characters string is of the form $s_1\ s_2\ s_3\ s_4$ with $s_1, s_2, s_3, s_4 \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}\}$. Based on the requirement, we have:

1. the number of possibility of letter at $s_1 : 4$,
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4. the number of possibility of letter at $s_4 : 1$.

Based on the product rule, the number of possible strings from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ is $4!$. Since $\mathbf{O}_1$ and $\mathbf{O}_2$ are not distinguishable, then using division rule, the number of different strings is:

Solutions: Problem 4 No.1

To make it easier, we first find the number of strings of four characters that we can have from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ (we differentiate the two letters of **O**). Notice that the four characters string is of the form $s_1\ s_2\ s_3\ s_4$ with $s_1, s_2, s_3, s_4 \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}\}$. Based on the requirement, we have:

1. the number of possibility of letter at $s_1 : 4$,
2. the number of possibility of letter at $s_2 : 3$,
3. the number of possibility of letter at $s_3 : 2$,
4. the number of possibility of letter at $s_4 : 1$.

Based on the product rule, the number of possible strings from the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}$ is $4!$. Since $\mathbf{O}_1$ and $\mathbf{O}_2$ are not distinguishable, then using division rule, the number of different strings is: $\frac{4!}{2!} = 12$ strings.

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{BO_1O_2K_1K_2E_1E_2PE_3R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}_1\mathbf{K}_2\mathbf{E}_1\mathbf{E}_2\mathbf{PE}_3\mathbf{R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

Based on the requirement and the product rule, the number of strings that we can have if we differentiate identical letters:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

- There are two **O**s, namely $\mathbf{O}_1$ and $\mathbf{O}_2$. Therefore, the number of permutations for **O**s is:

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}_1\mathbf{K}_2\mathbf{E}_1\mathbf{E}_2\mathbf{PE}_3\mathbf{R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

Based on the requirement and the product rule, the number of strings that we can have if we differentiate identical letters:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

- There are two **O**s, namely $\mathbf{O}_1$ and $\mathbf{O}_2$. Therefore, the number of permutations for **O**s is: $2!$ (namely $\mathbf{O}_1\mathbf{O}_2$ and $\mathbf{O}_2\mathbf{O}_1$).
- There are two **K**s, namely $\mathbf{K}_1$ and $\mathbf{K}_2$. Therefore, the number of permutations for **K**s is:

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{B}\mathbf{O}_1\mathbf{O}_2\mathbf{K}_1\mathbf{K}_2\mathbf{E}_1\mathbf{E}_2\mathbf{P}\mathbf{E}_3\mathbf{R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

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- There are two **O**s, namely $\mathbf{O}_1$ and $\mathbf{O}_2$. Therefore, the number of permutations for **O**s is: $2!$ (namely $\mathbf{O}_1\mathbf{O}_2$ and $\mathbf{O}_2\mathbf{O}_1$).
- There are two **K**s, namely $\mathbf{K}_1$ and $\mathbf{K}_2$. Therefore, the number of permutations for **K**s is: $2!$ (namely $\mathbf{K}_1\mathbf{K}_2$ and $\mathbf{K}_2\mathbf{K}_1$).
- There are three **E**s, namely $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$. Therefore, the number of permutations for **E**s is:

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}_1\mathbf{K}_2\mathbf{E}_1\mathbf{E}_2\mathbf{PE}_3\mathbf{R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

Based on the requirement and the product rule, the number of strings that we can have if we differentiate identical letters:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

- There are two **O**s, namely $\mathbf{O}_1$ and $\mathbf{O}_2$. Therefore, the number of permutations for **O**s is: $2!$ (namely $\mathbf{O}_1\mathbf{O}_2$ and $\mathbf{O}_2\mathbf{O}_1$).
- There are two **K**s, namely $\mathbf{K}_1$ and $\mathbf{K}_2$. Therefore, the number of permutations for **K**s is: $2!$ (namely $\mathbf{K}_1\mathbf{K}_2$ and $\mathbf{K}_2\mathbf{K}_1$).
- There are three **E**s, namely $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$. Therefore, the number of permutations for **E**s is: $3!$ (namely $\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3$, $\mathbf{E}_1\mathbf{E}_3\mathbf{E}_2$, $\mathbf{E}_2\mathbf{E}_1\mathbf{E}_3$, $\mathbf{E}_2\mathbf{E}_3\mathbf{E}_1$, $\mathbf{E}_3\mathbf{E}_1\mathbf{E}_2$, $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$).

Because the order of identical letters is ignored, then based on the division rule the value of $10!$ must be divided by product of $2!$, $2!$, and $3!$, so the number of strings that we can have is:

Solutions: Exercise 4 No. 2

To make it easier, we first find the number of strings that we can have form the word $\mathbf{BO}_1\mathbf{O}_2\mathbf{K}_1\mathbf{K}_2\mathbf{E}_1\mathbf{E}_2\mathbf{PE}_3\mathbf{R}$ (the identical letters are differentiated). Notice that the permutation of this string is of the form $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7\ s_8$ with $s_i \in \{\mathbf{B}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{K}_1, \mathbf{K}_2, \mathbf{E}_1, \mathbf{E}_2, \mathbf{P}, \mathbf{E}_3, \mathbf{R}\}$ for every $i = 1, \ldots, 8$.

Based on the requirement and the product rule, the number of strings that we can have if we differentiate identical letters:

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$

- There are two **O**s, namely $\mathbf{O}_1$ and $\mathbf{O}_2$. Therefore, the number of permutations for **O**s is: $2!$ (namely $\mathbf{O}_1\mathbf{O}_2$ and $\mathbf{O}_2\mathbf{O}_1$).
- There are two **K**s, namely $\mathbf{K}_1$ and $\mathbf{K}_2$. Therefore, the number of permutations for **K**s is: $2!$ (namely $\mathbf{K}_1\mathbf{K}_2$ and $\mathbf{K}_2\mathbf{K}_1$).
- There are three **E**s, namely $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$. Therefore, the number of permutations for **E**s is: $3!$ (namely $\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3$, $\mathbf{E}_1\mathbf{E}_3\mathbf{E}_2$, $\mathbf{E}_2\mathbf{E}_1\mathbf{E}_3$, $\mathbf{E}_2\mathbf{E}_3\mathbf{E}_1$, $\mathbf{E}_3\mathbf{E}_1\mathbf{E}_2$, $\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$).

Because the order of identical letters is ignored, then based on the division rule the value of $10!$ must be divided by product of $2!$, $2!$, and $3!$, so the number of strings that we can have is: $\frac{10!}{2!2!3!} = 151\,200$.

Problem 5

Exercise

1. Determine the number of subsets of $A = \{1, 2, 3, 4, 5, 6\}$ that contain exactly 3 members (example $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4\}$, and $\{4, 5, 6\}$).
2. Given a set $A = \{1, 2, 3, \ldots, n\}$. If $0 \leq k < n$, determine the number of subsets of A that contain exactly k members.

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B =$

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B = \{b_1, b_2, b_3\}$. Firstly, notice the way to obtain 3-ordered tuple (b_1, b_2, b_3) from $A = \{1, 2, 3, 4, 5, 6\}$. We have

1. the number of possibility for b_1 is

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B = \{b_1, b_2, b_3\}$. Firstly, notice the way to obtain 3-ordered tuple (b_1, b_2, b_3) from $A = \{1, 2, 3, 4, 5, 6\}$. We have

1. the number of possibility for b_1 is 6,
2. because $b_2 \neq b_1$, the number of possibility for b_2:

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B = \{b_1, b_2, b_3\}$. Firstly, notice the way to obtain 3-ordered tuple (b_1, b_2, b_3) from $A = \{1, 2, 3, 4, 5, 6\}$. We have

1. the number of possibility for b_1 is 6,
2. because $b_2 \neq b_1$, the number of possibility for b_2: 5,
3. because $b_3 \neq b_2 \neq b_1$, the number of possibility for b_3:

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B = \{b_1, b_2, b_3\}$. Firstly, notice the way to obtain 3-ordered tuple (b_1, b_2, b_3) from $A = \{1, 2, 3, 4, 5, 6\}$. We have

1. the number of possibility for b_1 is 6,
2. because $b_2 \neq b_1$, the number of possibility for b_2: 5,
3. because $b_3 \neq b_2 \neq b_1$, the number of possibility for b_3: 4.

Based on the product rule, the number of possibility for ordered tuple (b_1, b_2, b_3) is

Solutions: Exercise 5 No. 1

If $B \subseteq A$ and $|B| = 3$, we may assume $B = \{b_1, b_2, b_3\}$. Firstly, notice the way to obtain 3-ordered tuple (b_1, b_2, b_3) from $A = \{1, 2, 3, 4, 5, 6\}$. We have

1. the number of possibility for b_1 is 6,
2. because $b_2 \neq b_1$, the number of possibility for b_2: 5,
3. because $b_3 \neq b_2 \neq b_1$, the number of possibility for b_3: 4.

Based on the product rule, the number of possibility for ordered tuple (b_1, b_2, b_3) is $6 \cdot 5 \cdot 4$ possibilities.

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$.

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by 3 choices (one of the b_1, b_2, or b_3),
2. the second position can be filled by

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by 3 choices (one of the b_1, b_2, or b_3),
2. the second position can be filled by 2 choices,
3. the third position can be filled by

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by 3 choices (one of the b_1, b_2, or b_3),
2. the second position can be filled by 2 choices,
3. the third position can be filled by 1 choice.

Therefore, there are

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by 3 choices (one of the b_1, b_2, or b_3),
2. the second position can be filled by 2 choices,
3. the third position can be filled by 1 choice.

Therefore, there are $3 \cdot 2 \cdot 1 = 3!$ different orderings for b_1, b_2, and b_3. Based on the division rule, the number of different sets $\{b_1, b_2, b_3\} \subseteq A$ is

Because the order of elements in set is ignored, then $\{b_1, b_2, b_3\} = \{b_1, b_3, b_2\} = \cdots = \{b_3, b_2, b_1\}$. Therefore, the result $6 \cdot 5 \cdot 4$ must be divided by the number of ways of ordering for b_1, b_2, and b_3. The ordering of b_1, b_2, and b_3 can be done as follows:

1. the first position can be filled by 3 choices (one of the b_1, b_2, or b_3),
2. the second position can be filled by 2 choices,
3. the third position can be filled by 1 choice.

Therefore, there are $3 \cdot 2 \cdot 1 = 3!$ different orderings for b_1, b_2, and b_3. Based on the division rule, the number of different sets $\{b_1, b_2, b_3\} \subseteq A$ is $\frac{6 \cdot 5 \cdot 4}{3!} = 20$ sets.

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B =$$

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1:

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1: n,
- because $b_2 \neq b_1$, the number of possibilities for b_2:

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If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1: n,
- because $b_2 \neq b_1$, the number of possibilities for b_2: $n - 1$,
- because $b_3 \neq b_2 \neq b_1$, the number of possibilities for b_3:

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1: n,
- because $b_2 \neq b_1$, the number of possibilities for b_2: $n-1$,
- because $b_3 \neq b_2 \neq b_1$, the number of possibilities for b_3: $n-2$,
- $\vdots$
- because $b_k \neq b_i$ for every $i = 1, 2, \ldots, k-1$, the number of possibilities for b_k:

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1: n,
- because $b_2 \neq b_1$, the number of possibilities for b_2: $n-1$,
- because $b_3 \neq b_2 \neq b_1$, the number of possibilities for b_3: $n-2$,
- $\vdots$
- because $b_k \neq b_i$ for every $i = 1, 2, \ldots, k-1$, the number of possibilities for b_k: $n-k+1$.

Based on the product rule, the number of possibilities for ordered tuple $(b_1, b_2, \ldots, b_k)$:

$$(n)(n-1)\cdots(n-k+2)(n-k+1) =$$

Solutions Exercise 5 No. 2

If $B \subseteq A$ and $|B| = k \leq n$, we can assume B is as follows

$$B = \{b_1, b_2, \ldots, b_k\}.$$

First we seek the ways to obtain k-ordered tuple $(b_1, b_2, \ldots, b_k)$. We have

- the number of possibilities for b_1: n,
- because $b_2 \neq b_1$, the number of possibilities for b_2: $n-1$,
- because $b_3 \neq b_2 \neq b_1$, the number of possibilities for b_3: $n-2$,
- $\vdots$
- because $b_k \neq b_i$ for every $i = 1, 2, \ldots, k-1$, the number of possibilities for b_k: $n-k+1$.

Based on the product rule, the number of possibilities for ordered tuple $(b_1, b_2, \ldots, b_k)$:

$$(n)(n-1)\cdots(n-k+2)(n-k+1) = P(n,k).$$

Remember that for a set we ignore the order of elements because $\{b_1, b_2, \ldots, b_k\} = \cdots = \{b_k, b_{k-1}, \ldots, b_1\}$. Notice that there are $k!$ ways to sort $b_1, b_2, \ldots, b_k$, namely

- the first position can be filled by

Remember that for a set we ignore the order of elements because $\{b_1, b_2, \ldots, b_k\} = \cdots = \{b_k, b_{k-1}, \ldots, b_1\}$. Notice that there are $k!$ ways to sort $b_1, b_2, \ldots, b_k$, namely

- the first position can be filled by k choices (namely one of the b_1, b_2, ..., or b_k),
- the second position can be filled by

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- the second position can be filled by $k-1$ choices
- $\vdots$
- the k-th position can be filled by

Remember that for a set we ignore the order of elements because $\{b_1, b_2, \ldots, b_k\} = \cdots = \{b_k, b_{k-1}, \ldots, b_1\}$. Notice that there are $k!$ ways to sort $b_1, b_2, \ldots, b_k$, namely

- the first position can be filled by k choices (namely one of the b_1, b_2, ..., or b_k),
- the second position can be filled by $k-1$ choices
- $\vdots$
- the k-th position can be filled by 1 choice.

Based on the division rule, the number of different sets $\{b_1, b_2, \ldots, b_k\}$ is:

$$\frac{P(n,k)}{k!} =$$

Remember that for a set we ignore the order of elements because $\{b_1, b_2, \ldots, b_k\} = \cdots = \{b_k, b_{k-1}, \ldots, b_1\}$. Notice that there are $k!$ ways to sort $b_1, b_2, \ldots, b_k$, namely

- the first position can be filled by k choices (namely one of the b_1, b_2, ..., or b_k),
- the second position can be filled by $k-1$ choices
- $\vdots$
- the k-th position can be filled by 1 choice.

Based on the division rule, the number of different sets $\{b_1, b_2, \ldots, b_k\}$ is:

$$\frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}.$$

We know the last form as combination and it is denoted as $C(n,k)$ or C_k^n or ${}_nC_k$ or ${}^nC_k$ or $C_{n,k}$ or $\binom{n}{k}$.

Remember that for a set we ignore the order of elements because $\{b_1, b_2, \ldots, b_k\} = \cdots = \{b_k, b_{k-1}, \ldots, b_1\}$. Notice that there are $k!$ ways to sort $b_1, b_2, \ldots, b_k$, namely

- the first position can be filled by k choices (namely one of the b_1, b_2, ..., or b_k),
- the second position can be filled by $k-1$ choices
- $\vdots$
- the k-th position can be filled by 1 choice.

Based on the division rule, the number of different sets $\{b_1, b_2, \ldots, b_k\}$ is:

$$\frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}.$$

We know the last form as combination and it is denoted as $C(n,k)$ or C_k^n or ${}_nC_k$ or ${}^nC_k$ or $C_{n,k}$ or $\binom{n}{k}$. This means the number of ways to choose k elements of n elements $(n \geq k)$ by ignoring the order of the k elements .

Contents

Tree Diagram

Tree diagram can be used to solve combinatorics problems. Every branch denotes a possible choice and every path of the root (the root is the right-most node or the highest one) denotes the possible solution.

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How many binary string of length 4 that has no two consecutive digits of 1?

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How many binary string of length 4 that has no two consecutive digits of 1?

Answer: 8 string.

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How many binary string of length 4 that has no two consecutive digits of 1?
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