Simple Recurrence Relation and Its Solution

Discrete Mathematics – Second Term 2022-2023

MZI

School of Computing
Telkom University

SoC Tel-U

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications* , 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

Contents

Contents

A recursive structure is a natural structure in computer science.
GNU: GNU's not Unix

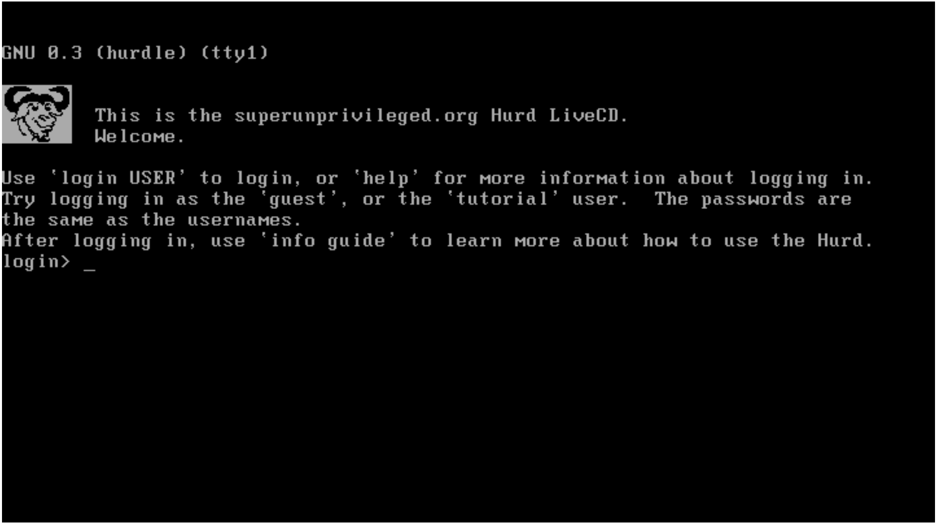

Image taken from Wikipedia.

Why Do We Need Recurrence Relation?

In computer science or daily life, many calculation problems can be modelled recursively. A mathematical expression is defined recursively if its definition refers to itself. A recursive problem can be modelled as a recurrence relation.

Example

Determine the number of binary strings (contain 0 or 1 only) of length n that has no two consecutive 0.

Illustration:

- Binary strings of length 1 that satisfy the condition:

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Illustration:

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- Binary strings of length 2 that satisfy the condition:

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- Binary strings of length 2 that satisfy the condition: 01, 10, and 11.
- Binary strings of length 3 that satisfy the condition:

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Determine the number of binary strings (contain 0 or 1 only) of length n that has no two consecutive 0.

Illustration:

- Binary strings of length 1 that satisfy the condition: 0 and 1
- Binary strings of length 2 that satisfy the condition: 01, 10, and 11.
- Binary strings of length 3 that satisfy the condition: 010, 011, 101, 110, and 111.

How many binary string of length n that satisfy the condition?

Example

In a system, a message always has a size of n kB with n is a nonnegative integer. The message is sent using an array with predetermined length defined as follows:

- If the size is 0 kB, then the array has length 1,
- If the size is 1 kB, then the array has length 2,
- If the size is n kB with $n > 1$, then the array has length of the length of array for $n-1$ kB message plus the length of array for $n-2$ kB message.

Determine the mathematical formula to determine the length of array that we need to send a message with the size of n kB. Furthermore, based on the formula, determine the length of array that we need to send a message of size 6 kB.

Example

Someone deposited his money with an interest rate of 7% per year. If this interest rate never change for 20 years and the money never been withdrawn, find the ratio of asset increment from the deposit (the final amount of money divided by the initial amount of money).

Contents

Some Important Definition

Definition (Recurrence relation)

A recurrence relation for a sequence (x_n) is an equation (formula) that defines the relation between x_n and one or more of its predecessor (namely $x_0, x_1, \ldots, x_{n-1}$) in the sequence for every $n \geq n_0$ where $n_0 \geq 1$.

Definition (Recurrence relation solution)

A sequence (x_n) is a solution to a recurrence relation if every term on that sequence satisfies the recurrence relation.

Definition (Recurrence Relation Initial Condition)

The preceding term(s) of x_n in a recurrence relation, namely $x_0, x_1, \ldots, x_{n-1}$, is called as *initial condition* of the corresponding recurrence relation.

Example

Let (x_n) be a sequence that satisfies the recurrence relation

$$x_n = x_{n-1} - x_{n-2}, \text{ for } n \geq 2, \tag{1}$$

with $x_0 = 3$ and $x_1 = 5$. Find x_2 and x_5.

Note that:

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3. From Equation (1), we have $x_2 =$

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4. x_5 can be found by using x_0 and x_1 alone , that is:

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$$x_5 = x_4 - x_3 =$$

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$$\begin{aligned} x_5 &= x_4 - x_3 = (x_3 - x_2) - (x_2 - x_1) \\ &= \end{aligned}$$

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Checking the Solution of Recurrence Relation

Example

Check whether the sequence (x_n) is a solution to recurrence relation

$$x_n = 2x_{n-1} - x_{n-2}, \text{ for } n \geq 2, \quad (2)$$

if

1. $x_n = 3n$
2. $x_n = 2^n$
3. $x_n = 5.$

Solution:

1. If $x_n = 3n$, then $x_{n-1} =$

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Solution:

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in other words equation (2) is satisfied.

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in other words equation (2) is satisfied. Thus, the sequence $(x_n) = (3n)$ is a solution to recurrence relation (2).

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$$2x_{n-1} - x_{n-2} = 2\left(2^{n-1}\right) - 2^{n-2} =$$

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in other words (2) is not satisfied. Hence, the sequence $(x_n) = (2^n)$ is not a solution to recurrence relation (2).

3. If $x_n = 5$, then $x_{n-1} =$

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3. If $x_n = 5$, then $x_{n-1} = 5$ and $x_{n-2} = 5$. We have

$$\begin{aligned} 2x_{n-1} - x_{n-2} &= 2 \cdot 5 - 5 \\ &= \end{aligned}$$

2. If $x_n = 2^n$ then $x_{n-1} = 2^{n-1}$ and $x_{n-2} = 2^{n-2}$. We have

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$$\begin{aligned} 2x_{n-1} - x_{n-2} &= 2 \cdot 5 - 5 \\ &= 5 = x_n, \end{aligned}$$

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in other words (2) is not satisfied. Hence, the sequence $(x_n) = (2^n)$ is not a solution to recurrence relation (2).

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Problem

Why do we have more than one solution for the recurrence relation (2)?

Contents

Investment Problem

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Someone deposited his money with an interest rate of 7% per year. If this interest rate never change for 20 years and the money never been withdrawn, find the ratio of asset increment from the deposit (the final amount of money divided by the initial amount of money).

Solution: suppose the initial amount of money is x_0 and the amount of money after n years is x_n. Therefore, there is a sequence x_n that satisfies the following recurrence relation:

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$$
\begin{aligned}
x_n &= x_{n-1} + 0.07x_{n-1}, \text{ for } n \geq 1, \text{ which equivalent to} \\
x_n &=
\end{aligned}
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Solution: suppose the initial amount of money is x_0 and the amount of money after n years is x_n. Therefore, there is a sequence x_n that satisfies the following recurrence relation:

$$\begin{aligned} x_n &= x_{n-1} + 0.07x_{n-1}, \text{ for } n \geq 1, \text{ which equivalent to} \\ x_n &= 1.07x_{n-1} \qquad (3) \end{aligned}$$

Thus,

$$x_1 =$$

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$$\begin{aligned} x_1 &= 1.07x_0 \\ x_2 &= \end{aligned}$$

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$$\begin{aligned} x_1 &= 1.07x_0 \\ x_2 &= 1.07x_1 = (1.07)^2\, x_0 \\ x_3 &= \end{aligned}$$

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&\vdots \\
x_n &=
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for $n = 20$, we have $x_{20} =$

Thus,

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for $n = 20$, we have $x_{20} = (1.07)^{20} x_0$. Then the ratio of asset increment of the deposit is

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x_3 &= 1.07x_2 = (1.07)^3\, x_0 \\
&\vdots \\
x_n &= 1.07x_{n-1} = (1.07)^n\, x_0,
\end{aligned}
$$

for $n = 20$, we have $x_{20} = (1.07)^{20}\, x_0$. Then the ratio of asset increment of the deposit is $\frac{x_{20}}{x_0} = (1.07)^{20}$.

Array Length Problem

Example

In a system, a message always has a size of n kB with n is a nonnegative integer. The message is sent using an array with predetermined length defined as follows:

- If the size is 0 kB, then the array has length 1,
- If the size is 1 kB, then the array has length 2,
- If the size is n kB with $n > 1$, then the array has length of the length of array for $n-1$ kB message plus the length of array for $n-2$ kB message.

Determine the mathematical formula to determine the length of array that we need for sending a message with the size of n kB. Furthermore, based on the formula, determine the length of array that we need for sending a message of size 6 kB.

Solution: Suppose L_n is the length of array required to send a message with the size of n kB, then we have:

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$L_0 = 1$, $L_1 = 2$, and $L_n =$

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$$L_0 = 1,\ L_1 = 2, \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

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Hence,

$$\begin{aligned} L_2 &= L_1 + L_0 = 2 + 1 = 3 \\ L_3 &= \end{aligned}$$

Solution: Suppose L_n is the length of array required to send a message with the size of n kB, then we have:

$L_0 = 1$, $L_1 = 2$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Hence,

$$\begin{aligned} L_2 &= L_1 + L_0 = 2 + 1 = 3 \\ L_3 &= L_2 + L_1 = 3 + 2 = 5 \\ L_4 &= \end{aligned}$$

Solution: Suppose L_n is the length of array required to send a message with the size of n kB, then we have:

$$L_0 = 1,\ L_1 = 2, \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

Hence,

$$\begin{aligned} L_2 &= L_1 + L_0 = 2 + 1 = 3 \\ L_3 &= L_2 + L_1 = 3 + 2 = 5 \\ L_4 &= L_3 + L_2 = 5 + 3 = 8 \\ L_5 &= \end{aligned}$$

Solution: Suppose L_n is the length of array required to send a message with the size of n kB, then we have:

$$L_0 = 1,\ L_1 = 2,\ \text{and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

Hence,

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\end{aligned}$$

We need array of length 21 to send a message with size 6 kB.

Problem

Is there an explicit formula for L_n?

Binary String Problem

Example

Define a recursive formula to determine the number of binary strings (that contain 0 or 1 only) of length n that has no two consecutive 0. Then based on the formula, find how many binary strings of length 5 that satisfies the requirement.

Illustration:

- Binary strings of length 1 that satisfy the condition:

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Illustration:

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- Binary strings of length 2 that satisfy the condition: 01, 10, and 11.
- Binary strings of length 3 that satisfy the condition: 010, 011, 101, 110, and 111.

Solution to Binary String Problem

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  α : any binary string of length $n-1$ which does not contain two consecutive 0s.

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  β : binary string of length $n-2$ which does not contain two consecutive 0s.

From case 1 and case 2, we have a recurrence relation $a_n =$

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We can also get the same number from the following steps:

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$$\begin{aligned} a_3 &= a_2 + a_1 = 3 + 2 = 5 \\ a_4 &= a_3 + a_2 = 5 + 3 = 8 \\ a_5 &= a_4 + a_2 = 8 + 5 = 13. \end{aligned}$$

Problem

Is there an explicit formula for a_n?

Contents

Linear Recurrence Relation

Definition

A linear recurrence relation with **constant coefficients** of degree k ($k \in \mathbb{N}$) for real number sequence $x_0, x_1, \ldots, x_n, \ldots$ is

$$a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} = f(n), \text{ for } k \leq n, \tag{4}$$

where $f(n)$ is a function, $a_0, a_1, \ldots, a_k$ are $k+1$ real numbers, $a_k \neq 0$.
If $f(n) = 0$, then the recurrence relation

$$a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} = 0, \text{ for } k \leq n, \tag{5}$$

is called **homogeneous** linear recurrence relation with constant coefficient. If $f(n) \neq 0$, then (4) is called **nonhomogeneous** linear recurrence relation with constant coefficient. Moreover, $x_n = c_n$ for $0 \leq n < k$ is the initial condition for (4) or (5).

Notice that homogeneous linear recurrence relation with constant coefficients of degree k can also be written as

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}$$

and nonhomogeneous linear recurrence relation with constant coefficients of degree k can also be written as

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k} + f(n),$$

for some nonzero function f .

Example

Recurrence relation:

1. $x_n = x_{n-1} + x_{n-2}$ is a homogeneous linear recurrence relation with constant coefficients of degree 2.
2. $2x_n + 5x_{n-1} = 2^n$ is a

Example

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1. $x_n = x_{n-1} + x_{n-2}$ is a homogeneous linear recurrence relation with constant coefficients of degree 2.
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4. $x_n = nx_{n-1} + x_{n-2}$ is a

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4. $x_n = nx_{n-1} + x_{n-2}$ is a homogeneous linear recurrence relation with non-constant coefficients of degree 2.
5. $3x_n = \frac{1}{n}x_{n-1} + x_{n-2}^n + x_{n-3} + n!$ is a

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Remark

The linearity of recurrence relation is similar to the linearity of linear equations in Matrices and Vector Spaces course.

Characteristic Polynomial

Definition

Let

$$a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} = f(n) \tag{6}$$

be a linear recurrence relation as defined in the previous section, the polynomial

$$p(\lambda) = a_0 \lambda^k + a_1 \lambda^{k-1} + \cdots + a_k$$

is a characteristic polynomial of recurrence relation (6). The equation $p(\lambda) = 0$ is called characteristic equation. The number r satisfies $p(r) = 0$ is called **characteristic root**. The number of occurrence of r as a root is called the multiplicity of the root.

Example

Determine the characteristic equation of the following recurrence relation:

1. $x_n = x_{n-1} + 2x_{n-2}$
2. $x_n = 6x_{n-1} - 9x_{n-2}$
3. $x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}$
4. $x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}$

Solution:

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Solution:

1. The recurrence relation can be rewritten as $x_n - x_{n-1} - 2x_{n-2} = 0$, so the corresponding characteristic equation is

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Solution to Recurrence Relation of Degree 2 (Different roots)

Theorem (Solution to recurrence relation of degree 2 (different roots))

Let $c_1, c_2 \in \mathbb{R}$ and equation $\lambda^2 - c_1\lambda - c_2 = 0$ has two different roots r_1 and r_2, then all solutions of the recurrence relation

$$x_n = c_1 x_{n-1} + c_2 x_{n-2}$$

has a form of

$$x_n = Ar_1^n + Br_2^n,\ n \in \mathbb{N}_0,$$

for some constants A and B.

Example

Example

Determine the solution of recurrence relation

$$x_n = x_{n-1} + 2x_{n-2}, \tag{7}$$

with initial condition $x_0 = 2$ and $x_1 = 7$.

Solution: Characteristic equation for the recurrence relation (7) is

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Solution: Characteristic equation for the recurrence relation (7) is

$$\lambda^2 - \lambda - 2 = 0 \text{ or } (\lambda + 1)(\lambda - 2) = 0$$

so the roots are

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Solution: Characteristic equation for the recurrence relation (7) is

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so the roots are $r_1 = -1$ and $r_2 = 2$. According to the previous theorem, the solution to recurrence relation (7) is

$$x_n =$$

Example

Example

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$$x_n = A \cdot 2^n + B \cdot (-1)^n ,$$

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observe that $x_0 = A + B = 2$, and $x_1 =$

Example

Example

Determine the solution of recurrence relation

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Solution: Characteristic equation for the recurrence relation (7) is

$$\lambda^2 - \lambda - 2 = 0 \text{ or } (\lambda + 1)(\lambda - 2) = 0$$

so the roots are $r_1 = -1$ and $r_2 = 2$. According to the previous theorem, the solution to recurrence relation (7) is

$$x_n = A \cdot 2^n + B \cdot (-1)^n,$$

observe that $x_0 = A + B = 2$, and $x_1 = 2A - B = 7$. Thus, we get $A =$

Example

Example

Determine the solution of recurrence relation

$$x_n = x_{n-1} + 2x_{n-2}, \tag{7}$$

with initial condition $x_0 = 2$ and $x_1 = 7$.

Solution: Characteristic equation for the recurrence relation (7) is

$$\lambda^2 - \lambda - 2 = 0 \text{ or } (\lambda + 1)(\lambda - 2) = 0$$

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$$x_n = A \cdot 2^n + B \cdot (-1)^n,$$

observe that $x_0 = A + B = 2$, and $x_1 = 2A - B = 7$. Thus, we get $A = 3$ and $B =$

Example

Example

Determine the solution of recurrence relation

$$x_n = x_{n-1} + 2x_{n-2}, \tag{7}$$

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$$x_n = A \cdot 2^n + B \cdot (-1)^n,$$

observe that $x_0 = A + B = 2$, and $x_1 = 2A - B = 7$. Thus, we get $A = 3$ and $B = -1$, so the general solution to the recurrence relation is

$$x_n = 3 \cdot 2^n - 1(-1)^n.$$

Solution to Recurrence Relation of Degree 2 (Twin Roots)

Theorem (Solution to recurrence relation of degree 2 (twin roots))

Let $c_1, c_2 \in \mathbb{R}$ with $c_2 \neq 0$ and equation $\lambda^2 - c_1\lambda - c_2 = 0$ has twin roots r_0, then all solutions of recurrence relation

$$x_n = c_1 x_{n-1} + c_2 x_{n-2}$$

has the form of

$$x_n = Ar_0^n + Bnr_0^n = (A + Bn)\, r_0^n,\ n \in \mathbb{N}_0,$$

for some constants A and B.

Example

Example

Determine the solution of recurrence relation

$$x_n = 6x_{n-1} - 9x_{n-2}, \tag{8}$$

with initial condition $x_0 = 1$ and $x_1 = 6$.

Solution: Characteristic equation for the recurrence relation (8) is

Example

Example

Determine the solution of recurrence relation

$$x_n = 6x_{n-1} - 9x_{n-2}, \tag{8}$$

with initial condition $x_0 = 1$ and $x_1 = 6$.

Solution: Characteristic equation for the recurrence relation (8) is

$$\lambda^2 - 6\lambda + 9 = 0 \text{ or } (\lambda - 3)^2 = 0$$

so we have the twin roots $r_0 =$

Example

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with initial condition $x_0 = 1$ and $x_1 = 6$.

Solution: Characteristic equation for the recurrence relation (8) is

$$\lambda^2 - 6\lambda + 9 = 0 \text{ or } (\lambda - 3)^2 = 0$$

so we have the twin roots $r_0 = 3$. According to the previous theorem, the solution to recurrence relation (8) is

$$x_n =$$

Example

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Determine the solution of recurrence relation

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$$\lambda^2 - 6\lambda + 9 = 0 \text{ or } (\lambda - 3)^2 = 0$$

so we have the twin roots $r_0 = 3$. According to the previous theorem, the solution to recurrence relation (8) is

$$x_n = A3^n + Bn3^n = (A + Bn)\,3^n,$$

observe that $x_0 =$

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observe that $x_0 = A = 1$, and $x_1 =$

Example

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$$x_n = A3^n + Bn3^n = (A + Bn)\, 3^n,$$

observe that $x_0 = A = 1$, and $x_1 = 3A + 3B = 6$. Thus, we have A

Example

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Determine the solution of recurrence relation

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$$x_n = A3^n + Bn3^n = (A + Bn)\,3^n,$$

observe that $x_0 = A = 1$, and $x_1 = 3A + 3B = 6$. Thus, we have $A = B = 1$, so the general solution is

$$x_n = (1 + n)\,3^n.$$

Supplement: Solution to Recurrence Relation of Degree k (Different Roots)

Theorem (Solution to recurrence relation of degree k (different roots))

Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and equation

$$\lambda^k - c_1\lambda^{k-1} - c_2\lambda^{k-2} - \cdots - c_k = 0$$

has k different roots $r_1, r_2, \ldots, r_k$, then the solution to recurrence relation

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$$

is

$$x_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n,\ n \in \mathbb{N}_0,$$

for some constants $A_1, A_2, \ldots, A_k$.

Example

Example

Find the solution to recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \tag{9}$$

with initial condition $x_0 = 2$, $x_1 = 5$, and $x_2 = 15$.

Solution: Characteristic equation for the recurrence relation (9) is

Example

Example

Find the solution to recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \tag{9}$$

with initial condition $x_0 = 2$, $x_1 = 5$, and $x_2 = 15$.

Solution: Characteristic equation for the recurrence relation (9) is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \text{ or}$$

Example

Example

Find the solution to recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \tag{9}$$

with initial condition $x_0 = 2$, $x_1 = 5$, and $x_2 = 15$.

Solution: Characteristic equation for the recurrence relation (9) is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \text{ or } (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0,$$

so the roots are

Example

Example

Find the solution to recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \qquad (9)$$

with initial condition $x_0 = 2$, $x_1 = 5$, and $x_2 = 15$.

Solution: Characteristic equation for the recurrence relation (9) is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \text{ or } (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0,$$

so the roots are $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. According to the previous theorem, the solution to recurrence relation (9) is

$$x_n =$$

Example

Example

Find the solution to recurrence relation

$$x_n = 6x_{n-1} - 11x_{n-2} + 6x_{n-3}, \tag{9}$$

with initial condition $x_0 = 2$, $x_1 = 5$, and $x_2 = 15$.

Solution: Characteristic equation for the recurrence relation (9) is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \text{ or } (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0,$$

so the roots are $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. According to the previous theorem, the solution to recurrence relation (9) is

$$x_n = A_1 1^n + A_2 2^n + A_3 3^n,$$

observe that:

1. $x_0 = A_1 + A_2 + A_3 = 2$,
2. $x_1 = A_1 + 2A_2 + 3A_3 = 5$,
3. $x_2 = A_1 + 4A_2 + 9A_3 = 15$,

so $A_1 =$

observe that:

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so $A_1 = 1$, $A_2 = -1$, and $A_3 =$

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so $A_1 = 1$, $A_2 = -1$, and $A_3 = 2$, then the general solution is

$$x_n = 1^n - 2^n + 2 \cdot 3^n.$$

Supplement: Solution to Recurrence Relation of Degree k (Duplicate Roots)

Theorem (Solution to recurrence relation of degree k (duplicate roots))

Let $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and equation

$$\lambda^k - c_1\lambda^{k-1} - c_2\lambda^{k-2} - \cdots - c_k = 0$$

has t different roots $(t < n)$, $r_1, r_2, \ldots, r_t$, each with multiplicity $m_1, m_2, \ldots, m_t$ $(m_1 + m_2 + \cdots + m_t = k)$, then the solution to recurrence relation

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$$

is of the form

$$\begin{aligned} x_n \quad = \quad & \left(A_{1,0} + A_{1,1}n + A_{1,2}n^2 + \cdots + A_{1,m_1-1}n^{m_1-1}\right) r_1^n \\ & + \left(A_{2,0} + A_{2,1}n + A_{2,2}n^2 + \cdots + A_{2,m_2-1}n^{m_2-1}\right) r_2^n \\ & + \cdots \\ & + \left(A_{t,0} + A_{t,1}n + A_{t,2}n^2 + \cdots + A_{t,m_t-1}n^{m_t-1}\right) r_t^n, \end{aligned}$$

Example

Example

Find the solution to recurrence relation

$$x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}, \tag{10}$$

with initial condition $x_0 = 1$, $x_1 = -2$, and $x_2 = -1$.

Solution: Characteristic equation for the recurrence relation (10) is

Example

Example

Find the solution to recurrence relation

$$x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}, \tag{10}$$

with initial condition $x_0 = 1$, $x_1 = -2$, and $x_2 = -1$.

Solution: Characteristic equation for the recurrence relation (10) is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \text{ or } (\lambda + 1)^3 = 0,$$

so the root is $r_1 =$

Example

Example

Find the solution to recurrence relation

$$x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}, \tag{10}$$

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Example

Example

Find the solution to recurrence relation

$$x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}, \tag{10}$$

with initial condition $x_0 = 1$, $x_1 = -2$, and $x_2 = -1$.

Solution: Characteristic equation for the recurrence relation (10) is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \text{ or } (\lambda + 1)^3 = 0,$$

so the root is $r_1 = -1$ with multiplicity $m_1 = 3$. According to the previous theorem, the solution to recurrence relation (10) is

$$x_n =$$

Example

Example

Find the solution to recurrence relation

$$x_n = -3x_{n-1} - 3x_{n-2} - x_{n-3}, \tag{10}$$

with initial condition $x_0 = 1$, $x_1 = -2$, and $x_2 = -1$.

Solution: Characteristic equation for the recurrence relation (10) is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \text{ or } (\lambda + 1)^3 = 0,$$

so the root is $r_1 = -1$ with multiplicity $m_1 = 3$. According to the previous theorem, the solution to recurrence relation (10) is

$$\begin{aligned} x_n &= \left(A_{1,0} + A_{1,1}n + A_{1,2}n^2\right) r_1^n \\ &= \left(A_{1,0} + A_{1,1}n + A_{1,2}n^2\right) (-1)^n \end{aligned}$$

observe that:

1. $x_0 = A_{1,0} = 1,$
2. $x_1 = -\left(A_{1,0} + A_{1,1} + A_{1,2}\right) = -2,$
3. $x_2 = A_{1,0} + 2A_{1,1} + 4A_{1,2} = -1,$

so $A_{1,0} =$

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3. $x_2 = A_{1,0} + 2A_{1,1} + 4A_{1,2} = -1,$

so $A_{1,0} = 1$, $A_{1,1} =$

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1. $x_0 = A_{1,0} = 1,$
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so $A_{1,0} = 1$, $A_{1,1} = 3$, and $A_{1,2} = -2$, then the general solution is

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so $A_{1,0} = 1$, $A_{1,1} = 3$, and $A_{1,2} = -2$, then the general solution is

$$x_n = \left(1 + 3n - 2n^2\right)(-1)^n.$$

Contents

Exercise: Solution to Homogeneous Linear Recurrence Relation

Exercise

Find the general solution to recurrence relation:

1. $w_n = 2w_{n-1}$ for every $n \geq 1$ with $w_0 = 3$. Write the value of w_{2023}.
2. $x_n = 4x_{n-2}$ for every $n \geq 2$ with $x_0 = 1$ and $x_1 = -1$. Write the value of x_{2023}.
3. $y_n = -2y_{n-1} - y_{n-2}$ for every $n \geq 2$ with $y_0 = 1$ and $y_1 = 4$. Write the value of y_{2023}.
4. $z_n = 3z_{n-1} - 2z_{n-2}$ for every $n \geq 2$ with $z_0 = 0$ and $z_1 = 1$. Write the value of z_{2023}.

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r =$

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r = 2$. According to the theorem, the solution is in the form of

$$w_n =$$

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r = 2$. According to the theorem, the solution is in the form of

$$w_n = Ar^n = A \cdot 2^n.$$

Because $w_0 =$

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r = 2$. According to the theorem, the solution is in the form of

$$w_n = Ar^n = A \cdot 2^n.$$

Because $w_0 = A \cdot 2^0 = A = 3$, then the solution is

$$w_n =$$

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r = 2$. According to the theorem, the solution is in the form of

$$w_n = Ar^n = A \cdot 2^n.$$

Because $w_0 = A \cdot 2^0 = A = 3$, then the solution is

$$w_n = 3 \cdot 2^n$$

and w_{2023} is

$$w_{2023} =$$

Solution to Problem 1

Characteristic equation for $w_n = 2w_{n-1}$ is $\lambda - 2 = 0$, so the root is $r = 2$. According to the theorem, the solution is in the form of

$$w_n = Ar^n = A \cdot 2^n.$$

Because $w_0 = A \cdot 2^0 = A = 3$, then the solution is

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and w_{2023} is

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Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 =$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 =$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n =$$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 =$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 =$

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$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A =$

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$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B =$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus, the solution is

$$x_n =$$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus, the solution is

$$x_n = \frac{3}{4} \cdot (-2)^n + \frac{1}{4} \cdot 2^n$$

and x_{2023} is

$$x_{2023} =$$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus, the solution is

$$x_n = \frac{3}{4} \cdot (-2)^n + \frac{1}{4} \cdot 2^n$$

and x_{2023} is

$$\begin{aligned} x_{2023} &= \frac{3}{4}(-2)^{2023} + \frac{1}{4}(2)^{2023} \\ &= \end{aligned}$$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus, the solution is

$$x_n = \frac{3}{4} \cdot (-2)^n + \frac{1}{4} \cdot 2^n$$

and x_{2023} is

$$\begin{aligned} x_{2023} &= \frac{3}{4}(-2)^{2023} + \frac{1}{4}(2)^{2023} \\ &= -\frac{3}{4}\left(2^{2023}\right) + \frac{1}{4}\left(2^{2023}\right) \\ &= \end{aligned}$$

Solution to Problem 2

Characteristic equation for $x_n = 4x_{n-2}$ is $\lambda^2 - 4 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 2) = 0$, the roots are $r_1 = -2$ and $r_2 = 2$. According to the theorem, the solution is in the form of

$$x_n = Ar_1^n + Br_2^n = A \cdot (-2)^n + B \cdot 2^n.$$

Because $x_0 = A + B = 1$ and $x_1 = -2A + 2B = -1$, so $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus, the solution is

$$x_n = \frac{3}{4} \cdot (-2)^n + \frac{1}{4} \cdot 2^n$$

and x_{2023} is

$$\begin{aligned} x_{2023} &= \frac{3}{4}(-2)^{2023} + \frac{1}{4}(2)^{2023} \\ &= -\frac{3}{4}\left(2^{2023}\right) + \frac{1}{4}\left(2^{2023}\right) \\ &= \left(-\frac{3}{4} + \frac{1}{4}\right)\left(2^{2023}\right) = -2^{2022}. \end{aligned}$$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 =$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

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$$y_n = Ar_0^n + Bnr_0^n = (A + Bn)\, r_0^n = (A + Bn)\, (-1)^n\,.$$

Because $y_0 =$

Solution to Problem 3

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$$y_n = Ar_0^n + Bnr_0^n = (A + Bn)\, r_0^n = (A + Bn)\,(-1)^n\,.$$

Because $y_0 = A\,(-1)^0 = 1$ and $y_1 =$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

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Because $y_0 = A\,(-1)^0 = 1$ and $y_1 = (A + B)\,(-1) = 4$, then $A =$

Solution to Problem 3

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Because $y_0 = A\,(-1)^0 = 1$ and $y_1 = (A + B)\,(-1) = 4$, then $A = 1$ and $B =$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

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Because $y_0 = A\,(-1)^0 = 1$ and $y_1 = (A + B)\,(-1) = 4$, then $A = 1$ and $B = -5$. Hence, the solution is

$$y_n =$$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

$$y_n = Ar_0^n + Bnr_0^n = (A + Bn)\, r_0^n = (A + Bn)\,(-1)^n\,.$$

Because $y_0 = A\,(-1)^0 = 1$ and $y_1 = (A + B)\,(-1) = 4$, then $A = 1$ and $B = -5$. Hence, the solution is

$$y_n = (1 - 5n)\,(-1)^n$$

and y_{2023} is

$$y_{2023} \quad =$$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

$$y_n = Ar_0^n + Bnr_0^n = (A + Bn)\, r_0^n = (A + Bn)\,(-1)^n .$$

Because $y_0 = A\,(-1)^0 = 1$ and $y_1 = (A + B)\,(-1) = 4$, then $A = 1$ and $B = -5$. Hence, the solution is

$$y_n = (1 - 5n)\,(-1)^n$$

and y_{2023} is

$$\begin{aligned} y_{2023} &= (1 - 5\,(2023))\,(-1)^{2023} \\ &= \end{aligned}$$

Solution to Problem 3

Characteristic equation for $y_n = -2y_{n-1} - y_{n-2}$ is $\lambda^2 + 2\lambda + 1 = 0 \Leftrightarrow (\lambda + 1)^2 = 0$, the roots are $r_0 = -1$. According to the theorem, solution is in the form of

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$$y_n = (1 - 5n)\,(-1)^n$$

and y_{2023} is

$$\begin{aligned} y_{2023} &= (1 - 5\,(2023))\,(-1)^{2023} \\ &= 5\,(2023) - 1 \\ &= 10\,114. \end{aligned}$$

Solution to Problem 4

Characteristic equation for $z_n = 3z_{n-1} - 2z_{n-2}$ is

Solution to Problem 4

Characteristic equation for $z_n = 3z_{n-1} - 2z_{n-2}$ is
$\lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda - 1)(\lambda - 2) = 0$, the roots are $r_1 =$

Solution to Problem 4

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According to the theorem, the solution is in form of

$$z_n = Ar_1^n + Br_2^n = A(1)^n + B(2)^n = A + B(2)^n .$$

Because $z_0 =$

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Because $z_0 = A + B(2)^0 = A + B = 0$ and $z_1 =$

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Because $z_0 = A + B(2)^0 = A + B = 0$ and $z_1 = A + B(2)^1 = A + 2B = 1$, then $A = -1$ and $B = 1$. Thus, the solution is

$$z_n = -1 + 2^n$$

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and z_{2023} is

$$z_{2023} = -1 + 2^{2023} = 2^{2023} - 1.$$

Contents

Challenging Problem

Exercise

Determine the explicit solution of the following recurrence relations:

1. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$,
2. $a_n = 7a_{n-2} - 6a_{n-3}$ for $n \geq 3$ with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$

Contents

Homogeneous Recurrence Relation Corresponded to Nonhomogeneous Recurrence Relation

The course material related with solution of recurrence relation linear non homogeneous with constant coefficient will be studied further in Analysis of Algorithm Course.

Definition (Homogeneous Recurrence Relation Corresponded to Nonhomogeneous Recurrence Relation)

Let

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + f(n), \quad (11)$$

with constants c_i for every $i \in \{1, \ldots, n-k\}$ and f is a nonzero function, then

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} \quad (12)$$

is homogeneous recurrence relation that corresponds to nonhomogeneous recurrence relation(11).

Homogeneous Solution and Particular Solution

Theorem

Suppose a sequence $\left(x_n^{(h)}\right)$ is a general solution to homogeneous linear recurrence relation with constant coefficient

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} \tag{13}$$

and $\left(x_n^{(p)}\right)$ is a sequence that satisfies nonhomogeneous linear recurrence relation with constant coefficient

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + f(n), \tag{14}$$

then every solution of nonhomogeneous linear recurrence relation (14) is the sequence

$$\left(x_n^{(h)} + x_n^{(p)}\right).$$

The sequence $\left(x_n^{(p)}\right)$ is called as a particular solution for recurrence relation (14) and $\left(x_n^{(h)}\right)$ is called a homogeneous solution for recurrence relation (13).

Theorem

If the sequence (u_n) is a particular solution to nonhomogeneous linear recurrence relation

$$c_0 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k} = f(n), \tag{15}$$

for some $k \leq n$, and the sequence (v_n) is a particular solution to nonhomogeneous linear recurrence relation

$$c_0 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k} = g(n), \tag{16}$$

for some $k \leq n$, then

$$(w_n) = (u_n + v_n) \tag{17}$$

is a particular solution to nonhomogeneous linear recurrence relation

$$c_0 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k} = f(n) + g(n). \tag{18}$$

How to Find Particular Solution?

The method to find the particular solution $\left(x_n^{(p)}\right)$ depends on $f(n)$ as follows.

1. If $f(n)$ in linear recurrence relation (14) is polynomial

$$f(n) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_k x^k = \sum_{i=0}^{k} \alpha_i x^i,$$

then the corresponding particular solution $\left(x_n^{(p)}\right)$ has a similar form, which is

$$\left(x_n^{(p)}\right) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k = \sum_{i=0}^{k} \beta_i x^i. \tag{19}$$

Coefficients β_i for $i \in \{0, \ldots, k\}$ can be found by substituting (19) to (14).

2. If $f(n)$ in linear recurrence relation (14) is

$$f(n) = d^n \sum_{i=0}^{k} \alpha_i x^i,$$

for some constants d, then the corresponding particular solution $\left(x_n^{(p)}\right)$ also has a similar form, which is

$$f(n) = d^n \sum_{i=0}^{k} \beta_i x. \qquad (20)$$

Coefficients β_i for $i \in \{0, \ldots, k\}$ can be found by substituting (20) to (14).

Example

Example

Find all solutions to recurrence relation

$$x_n = 3x_{n-1} + 2n. \tag{21}$$

Recurrence relation (21) is a nonhomogeneous linear recurrence relation with $f(n) =$

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Recurrence relation (21) is a nonhomogeneous linear recurrence relation with $f(n) = 2n$. Because $f(n)$ is polynomial of degree 1, then we take

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Recurrence relation (21) is a nonhomogeneous linear recurrence relation with $f(n) = 2n$. Because $f(n)$ is polynomial of degree 1, then we take

$$p_n = An + B, \text{ with } A \text{ and } B \text{ some constants}$$

to get the particular solution of (21). By substituting p_n to (21) we have

$$p_n =$$

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to get the particular solution of (21). By substituting p_n to (21) we have

$$\begin{aligned} p_n &= 3p_{n-1} + 2n \\ An + B &= 3(A(n-1) + B) + 2n \end{aligned}$$

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$$\begin{aligned} p_n &= 3p_{n-1} + 2n \\ An + B &= 3(A(n-1) + B) + 2n \\ (-2A - 2)n + (3A - 2B) &= 0, \end{aligned}$$

so $A =$

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$$\begin{aligned} p_n &= 3p_{n-1} + 2n \\ An + B &= 3\left(A(n-1) + B\right) + 2n \\ (-2A - 2)\,n + (3A - 2B) &= 0, \end{aligned}$$

so $A = -1$ and $B = -\frac{3}{2}$. Thus, the particular solution is $x_n^{(p)} =$

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$$p_n = An + B, \text{ with } A \text{ and } B \text{ some constants}$$

to get the particular solution of (21). By substituting p_n to (21) we have

$$\begin{aligned} p_n &= 3p_{n-1} + 2n \\ An + B &= 3\left(A\left(n-1\right) + B\right) + 2n \\ \left(-2A - 2\right)n + \left(3A - 2B\right) &= 0, \end{aligned}$$

so $A = -1$ and $B = -\frac{3}{2}$. Thus, the particular solution is $x_n^{(p)} = -n - \frac{3}{2}$.

Homogeneous solution to homogeneous recurrence relation that corresponds to (21)

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$$x_n = 3x_{n-1},$$

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is $x_n^{(h)} = C \cdot 3^n$, for some constant C. So, by the theorem in previous section, the general solution to (21) is

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Homogeneous solution to homogeneous recurrence relation that corresponds to (21)

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is $x_n^{(h)} = C \cdot 3^n$, for some constant C. So, by the theorem in previous section, the general solution to (21) is

$$\begin{aligned} x_n &= x_n^{(h)} + x_n^{(p)} \\ &= \end{aligned}$$

Homogeneous solution to homogeneous recurrence relation that corresponds to (21)

$$x_n = 3x_{n-1},$$

is $x_n^{(h)} = C \cdot 3^n$, for some constant C. So, by the theorem in previous section, the general solution to (21) is

$$\begin{aligned} x_n &= x_n^{(h)} + x_n^{(p)} \\ &= C \cdot 3^n - n - \frac{3}{2}. \end{aligned}$$

Example

Find all solutions to recurrence relation

$$x_n = 5x_{n-1} - 6x_{n-2} + 7^n. \tag{22}$$

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$$x_n = 5x_{n-1} - 6x_{n-2} + 7^n. \tag{22}$$

Homogeneous solution of recurrence relation (22) is

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Example

Find all solutions to recurrence relation

$$x_n = 5x_{n-1} - 6x_{n-2} + 7^n. \tag{22}$$

Homogeneous solution of recurrence relation (22) is

$$x_n^{(h)} = A \cdot 3^n + B \cdot 2^n,$$

for some constants A and B. Because (22) is nonhomogeneous linear recurrence relation with $f(n) =$

Example

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for some constants A and B. Because (22) is nonhomogeneous linear recurrence relation with $f(n) = 7^n$, a particular solution we can try is

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Homogeneous solution of recurrence relation (22) is

$$x_n^{(h)} = A \cdot 3^n + B \cdot 2^n,$$

for some constants A and B. Because (22) is nonhomogeneous linear recurrence relation with $f(n) = 7^n$, a particular solution we can try is

$$x_n^{(p)} = \alpha \cdot 7^n, \text{ for some constant } \alpha.$$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned} \alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\ & \text{multiply both sides by } 7^2 \text{ to get} \end{aligned}$$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned}
\alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\
&\quad \text{multiply both sides by } 7^2 \text{ to get} \\
49\alpha \cdot 7^n &= 35\alpha \cdot 7^n - 6\alpha \cdot 7^n + 49 \cdot 7^n
\end{aligned}$$

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49\alpha \cdot 7^n &= 35\alpha \cdot 7^n - 6\alpha \cdot 7^n + 49 \cdot 7^n \\
20\alpha \cdot 7^n &= 49 \cdot 7^n
\end{aligned}$$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned}
\alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\
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49\alpha \cdot 7^n &= 35\alpha \cdot 7^n - 6\alpha \cdot 7^n + 49 \cdot 7^n \\
20\alpha \cdot 7^n &= 49 \cdot 7^n \\
\alpha &= \frac{49}{20}.
\end{aligned}$$

Thus, $x_n^{(p)} =$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned}
\alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\
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20\alpha \cdot 7^n &= 49 \cdot 7^n \\
\alpha &= \frac{49}{20}.
\end{aligned}$$

Thus, $x_n^{(p)} = \frac{49}{20} \cdot 7^n$. We have the general solution to (22) is

$$x_n =$$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned}
\alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\
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49\alpha \cdot 7^n &= 35\alpha \cdot 7^n - 6\alpha \cdot 7^n + 49 \cdot 7^n \\
20\alpha \cdot 7^n &= 49 \cdot 7^n \\
\alpha &= \frac{49}{20}.
\end{aligned}$$

Thus, $x_n^{(p)} = \frac{49}{20} \cdot 7^n$. We have the general solution to (22) is

$$\begin{aligned}
x_n &= x_n^{(h)} + x_n^{(p)} \\
&=
\end{aligned}$$

By substituting $x_n^{(p)}$ to (22), we have that

$$\begin{aligned}
\alpha \cdot 7^n &= 5\alpha \cdot 7^{n-1} - 6\alpha \cdot 7^{n-2} + 7^n, \\
& \quad \text{multiply both sides by } 7^2 \text{ to get} \\
49\alpha \cdot 7^n &= 35\alpha \cdot 7^n - 6\alpha \cdot 7^n + 49 \cdot 7^n \\
20\alpha \cdot 7^n &= 49 \cdot 7^n \\
\alpha &= \frac{49}{20}.
\end{aligned}$$

Thus, $x_n^{(p)} = \frac{49}{20} \cdot 7^n$. We have the general solution to (22) is

$$\begin{aligned}
x_n &= x_n^{(h)} + x_n^{(p)} \\
&= A \cdot 3^n + B \cdot 2^n + \frac{49}{20} \cdot 7^n.
\end{aligned}$$

Theorem Related to Particular Solution

Theorem

Suppose the sequence (x_n) satisfies nonhomogeneous linear recurrence relation

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + f(n), \tag{23}$$

with c_i $(i = 1, 2, \ldots, k)$ is a real number and

$$f(n) = \left(d_0 + d_1 n + d_2 n^2 + \cdots + d_t n^t\right) s^n,$$

with d_i $(i = 1, 2, \ldots, k)$ and s are real numbers, then

1. if s is not a root of characteristic equation of homogeneous recurrence relation that corresponds to (23) then there is a particular solution in the form of
$$\left(A_0 + A_1 n + A_2 n^2 + \cdots + A_{t-1} n^{t-1} + A_t n^t\right) s^n,$$
with $A_0, \ldots, A_t \in \mathbb{R}$.

2. if s a root with multiplicity m from characteristic equation of homogeneous recurrence relation corresponded to (23) then there is a particular solution in the form of

$$n^m \left(A_0 + A_1 n + A_2 n^2 + \cdots + A_{t-1} n^{t-1} + A_t n^t\right) s^n,$$

with $A_0, \ldots, A_t \in \mathbb{R}$.

Exercise

Exercise

Find the possible particular solution of nonhomogeneous linear recurrence relation

$$x_n = 6x_{n-1} - 9x_{n-2} + f(n), \tag{24}$$

if

1. $f(n) = 3^n$,
2. $f(n) = n \cdot 3^n$,
3. $f(n) = n^2 \cdot 2^n$, and
4. $f(n) = \left(n^2 + 1\right) \cdot 3^n$.

Homogeneous recurrence relation that corresponds to (24) is

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$$x_n^{(h)} = (A + Bn) \cdot 3^n.$$

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1. For $f(n) = 3^n$, because 3 is a characteristic root with multiplicity 2 for homogeneous linear recurrence relation (25), then according to the theorem, the possible particular solution is of the form $x_n^{(p)} = n^2 (A_0) \cdot 3^n$.
2. For $f(n) = n \cdot 3^n$, because 3 is a characteristic root with multiplicity 2 for homogeneous linear recurrence relation (25), then according to the theorem, the possible particular solution is of the form

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2. For $f(n) = n \cdot 3^n$, because 3 is a characteristic root with multiplicity 2 for homogeneous linear recurrence relation (25), then according to the theorem, the possible particular solution is of the form $x_n^{(p)} = n^2 (A_0 + A_1 n) \cdot 3^n$.
3. For $f(n) = n^2 \cdot 2^n$, because 2 is not a characteristic root for homogeneous linear recurrence relation (25), then according to the theorem, the possible particular solution is of the form

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4. For $f(n) = \left(n^2 + 1\right) \cdot 3^n$, because 3 is a characteristic root with multiplicity 2 for homogeneous linear recurrence relation (25), then according to the theorem, the possible particular solution is of the form $x_n^{(p)} = n^2 \left(A_0 + A_1 n + A_2 n^2\right) \cdot 3^n$.