Function: Definitions, Properties, and Representations

Discrete Mathematics – Second Term 2022-2023

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications* , 5th Edition., 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012) at Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret at Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

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Contents

Definition

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Given two nonempty sets A and B. A function from A to B is a **relation** that associates every member of A into exactly one member of B. A function from A to B can be written using the following notation

$$\begin{aligned} f &: A \to B \\ &: a \mapsto b, \text{ with } a \in A \text{ and } b \in B \end{aligned}$$

A function is also called as a mapping or a transformation. The notation $f(a) = b$ means that a is mapped (by f) to b.
The set A is called as a domain of f and written as $\operatorname{dom}(f)$, while set B is called as codomain of f and is written as $\operatorname{cod}(f)$.

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- A partial function is a function without *total property*, a partial function $f : A \rightarrow B$ is a function with the following property: f associates each member of A with at most one member of B. We also have seen an example of a partial function in high school as well as Calculus, such as $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$.

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- A partial function is a function without *total property*, a partial function $f : A \rightarrow B$ is a function with the following property: f associates each member of A with at most one member of B. We also have seen an example of a partial function in high school as well as Calculus, such as $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$. Notice that $\operatorname{dom}(f) \neq \mathbb{R}$ because f is undefined for $x < 0$, for example, the value of $f(-3)$ is undefined.

Image (Map), Preimage (Pre-map), and Range

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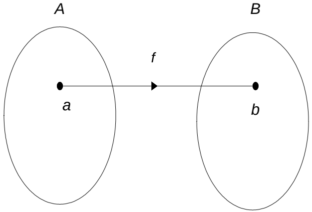

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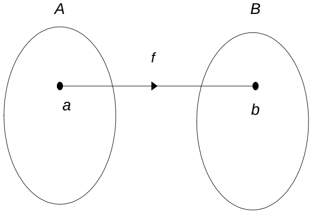

A *range* of f, denoted as $\operatorname{ran}(f)$ or $\operatorname{Im}(f)$, is defined as $\operatorname{ran}(f) = \operatorname{Im}(f) = \{b \in B \mid b = f(a)$, for an $a \in A\}$. It is obvious that $\operatorname{ran}(f) \subseteq \operatorname{cod}(f)$.

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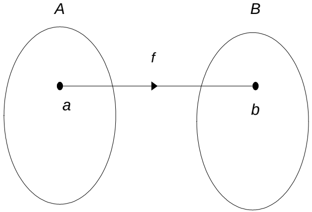

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If f is a function from A to B, we say that f maps A to B.

Equality of Two Functions

Definition

Two functions f and g are **equal** if

1. $\mathrm{dom}\,(f) = \mathrm{dom}\,(g)$
2. $\mathrm{cod}\,(f) = \mathrm{cod}\,(g)$
3. for every x in domain, $f\,(x) = g\,(x)$.

We consider the equality of two functions as the equality of sets (regarding a function as a relation).

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A function $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x + 1$ and $g : \mathbb{Q} \to \mathbb{Q}$ where $g(x) = x + 1$ is not equal, although they have the same formula.

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- A function is also a relation, therefore the properties of relation are applied on function.

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- an arrow diagram (if the domain and codomain of the function have finite cardinality)
- a matrix $0 - 1$ (if the domain and codomain of the function have finite cardinality)
- a digraph (if the domain and codomain of the function are equal and have a finite cardinality)

We have already seen the representation of ordered pair, arrow diagram, matrix, and digraph in the course material about relation.

Function as Ordered Pairs

As in a relation, a function can be represented as an ordered pair.

Example

A relation $f = \{(1, a), (2, b), (3, c)\}$ from set $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is a function.

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A relation $g = \{(1, a), (2, b), (3, b)\}$ from set $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is a function. We can write this function as $g(1) = a$, $g(2) = b$, and $g(3) = b$. We have $\mathrm{dom}(g) = X$, $\mathrm{cod}(g) = Y$, and $\mathrm{ran}(g) = \mathrm{Im}(g) = \{a, b\} \subset Y$.

Exercise: Function as Ordered Pairs

Exercise

Determine whether each of these relations is a function or not. If it is a function, determine its domain, codomain, and range.

1. f is a relation from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ where $f = \{(1, a), (2, a), (3, a)\}$.
2. g is a relation from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ where $g = \{(1, a), (2, b), (2, c), (3, c)\}$.
3. h is a relation from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ where $h = \{(1, a), (2, c)\}$.
4. k is a relation from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ where $k = \{(1, a), (2, b), (2, c)\}$.

Solution:

1. f is a function from X to Y where $\mathrm{dom}\left(f\right) = X$, $\mathrm{cod}\left(f\right) = Y$, $\mathrm{ran}\left(f\right) = \mathrm{Im}\left(f\right) = \{a\}$.

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Function Representation with Assignment Formula

The most usual way to represent a function is by assignment formula.

Example

Let $f, g, h : \mathbb{Z} \to \mathbb{Z}$ be relations defined as below:

1. $f(x) = x + 1$
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Determine whether each of these relations is a function or not. If it is a function, determine its domain, codomain, and range.

1. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$.
2. $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = \frac{1}{x}$.
3. $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ where $h(x) = \frac{1}{x}$.
4. $k : \mathbb{Q}^+ \to \mathbb{Q}^+$ where $k(x) = \sqrt{x}$.

Solution:

1. $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $f(x) = x^2$ is a function, where $\mathrm{dom}(f) = \mathbb{Z}$, $\mathrm{cod}(f) = \mathbb{Z}$, and $\mathrm{ran}(f) = \mathrm{Im}(f) =$

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2. $g : \mathbb{Z} \rightarrow \mathbb{Z}$ with $g(x) = \frac{1}{x}$ is not a function, because $g(0)$ is undefined. g is a partial function, because if $x = 1$ or $x = -1$, so the value of $g(x)$ is defined and has a single value.

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1. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x^2$ is a function, where $\mathrm{dom}(f) = \mathbb{Z}$, $\mathrm{cod}(f) = \mathbb{Z}$, and $\mathrm{ran}(f) = \mathrm{Im}(f) = \{y \in \mathbb{Z} \mid y = x^2 \text{ for an } x \in \mathbb{Z}\} = \{x^2 \mid x \in \mathbb{Z}\}$. Therefore, $\mathrm{ran}(f)$ or $\mathrm{Im}(f)$ is a set of all non-negative integers that are perfect squares.
2. $g : \mathbb{Z} \to \mathbb{Z}$ with $g(x) = \frac{1}{x}$ is not a function, because $g(0)$ is undefined. g is a partial function, because if $x = 1$ or $x = -1$, so the value of $g(x)$ is defined and has a single value.
3. $h : \mathbb{Q}^+ \to \mathbb{Q}^+$ with $h(x) = \frac{1}{x}$ is a function, where $\mathrm{dom}(h) = \mathbb{Q}^+$, $\mathrm{cod}(h) = \mathbb{Q}^+$, and $\mathrm{ran}(h) = \mathrm{Im}(h) = \{y \in \mathbb{Q}^+ \mid y = \frac{1}{x} \text{ for an } x \in \mathbb{Q}^+\} = \mathbb{Q}^+$, because for every $y \in \mathbb{Q}^+$ there is $x \in \mathbb{Q}^+$ such that $xy = 1$. Therefore, $\mathrm{ran}(h)$ or $\mathrm{Im}(h)$ is $\mathbb{Q}^+$.
4. $k : \mathbb{Q}^+ \to \mathbb{Q}^+$ with $k(x) = \sqrt{x}$ is not a function, because $k(2)$ is undefined. This happens because $k(2) = \sqrt{2} \notin \mathbb{Q}^+$ (remember that $\sqrt{2}$ is an irrational number). k is a partial function, because if $\sqrt{x}$ is defined and $\sqrt{x} \in \mathbb{Q}^+$, then it has a single value.

Function Representation in Natural Language

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ with $f(x) = x^2$. Then f can be described in natural language as: "f maps each integer to its square".

We can see that it is simpler to write a function in assignment formula, rather than natural language, but it is not always the case.

Example

Let $A = \{x \mid x$ is a string of length 5 whose characters are in $\{0, 1, 2\}\}$. A function $f : A \to \mathbb{N}_0$ is defined as the number of character of 2 within a string x. For example:

1. $f(21222) =$

We can see that it is simpler to write a function in assignment formula, rather than natural language, but it is not always the case.

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Let $A = \{x \mid x \text{ is a string of length } 5 \text{ whose characters are in } \{0,1,2\}\}$. A function $f : A \to \mathbb{N}_0$ is defined as the number of character of 2 within a string x. For example:

1. $f(21222) = 4$
2. $f(21202) =$

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f can also be written in assignment formula representation.

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3. $f(02102) = 2$.

f can also be written in assignment formula representation. Let $x = x_1x_2x_3x_4x_5$

$$f(x) = f(x_1x_2x_3x_4x_5) =$$

We can see that it is simpler to write a function in assignment formula, rather than natural language, but it is not always the case.

Example

Let $A = \{x \mid x$ is a string of length 5 whose characters are in $\{0, 1, 2\}\}$. A function $f : A \to \mathbb{N}_0$ is defined as the number of character of 2 within a string x. For example:

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$$f(x) = f(x_1x_2x_3x_4x_5) = \left|\{x_i \mid (x_i = 2) \land (1 \leq i \leq 5)\}\right|.$$

Function Representation in Programming Language

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function with $f(x) = \begin{cases} 3x+1, & x \text{ is odd} \\ \frac{x}{2}, & x \text{ is even} \end{cases}$. This function can be written in Python language as follows:

Function f in Python

```
def f(x):
    if (x%2 == 1):
        return (3 * x + 1)
    else:
        return (x // 2)
```

Contents

Contents

Injective Function

Definition (Injective function)

Let $f : A \rightarrow B$ be a function, f is injective (one-to-one) if every element in the domain of f is mapped to a different element in B, or in other words, for every $x_1, x_2 \in \mathrm{dom}\,(f)$ we have: if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$; we write it in predicate logic as follows:

$$(\forall x_1)(\forall x_2)\ \ (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))\text{, which is equivalent to}$$
$$(\forall x_1)(\forall x_2)\ \ (f(x_1) = f(x_2) \rightarrow x_1 = x_2)\,.$$

If f is an injective function, then f is also called an injection.

Remark

Note that $f : A \rightarrow B$ is injective if **there is no two different elements** in A **that has the same image**.

Examples of Injective Function

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$. A function $f : A \to B$ defined as

$$f(a) = 1,\ f(b) = 3,\ f(c) = 5, \text{ and } f(d) = 2$$

is an injective function, because there is no two elements in A with the same image. We have: if $x \neq y$ then $f(x) \neq f(y)$. An arrow diagram from this function can be described as follows

Examples of Injective Function

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$. A function $f : A \to B$ defined as

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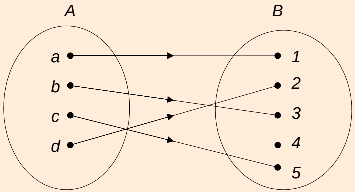

Checking the Injectivity of a Function

1. To prove f is injective, we show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
2. To prove f is not injective, we must find $x_1, x_2 \in \mathrm{dom}(f)$ with $x_1 \neq x_2$ that satisfies $f(x_1) = f(x_2)$.

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
2. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$, and $f = \{(1, u), (2, u), (3, v)\}$.
3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
2. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$, and $f = \{(1, u), (2, u), (3, v)\}$.
3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$,

Exercise

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Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
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3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
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Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.
2. f is not injective, because $1 \neq 2$ but $f(1) = f(2) = u$.

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \rightarrow B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
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3. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \rightarrow \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.
2. f is not injective, because $1 \neq 2$ but $f(1) = f(2) = u$.
3. f is not injective, because $-1 \neq 1$ but $f(-1) = f(1) = 2$.

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
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3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.
2. f is not injective, because $1 \neq 2$ but $f(1) = f(2) = u$.
3. f is not injective, because $-1 \neq 1$ but $f(-1) = f(1) = 2$.
4. f is injective, because we have:
   $f(x_1) = f(x_2) \Rightarrow$

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
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3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.
2. f is not injective, because $1 \neq 2$ but $f(1) = f(2) = u$.
3. f is not injective, because $-1 \neq 1$ but $f(-1) = f(1) = 2$.
4. f is injective, because we have:
   $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow$

Exercise

Exercise

Check whether the following functions are injective:

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$, and $f = \{(1, w), (2, u), (3, v)\}$.
2. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$, and $f = \{(1, u), (2, u), (3, v)\}$.
3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is injective, because $f(1) = w$, $f(2) = u$, and $f(3) = v$, there is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ and $f(a_1) = f(a_2)$.
2. f is not injective, because $1 \neq 2$ but $f(1) = f(2) = u$.
3. f is not injective, because $-1 \neq 1$ but $f(-1) = f(1) = 2$.
4. f is injective, because we have:
   $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. So,
   $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Contents

Surjective Function

Definition (Surjective function)

Let $f : A \to B$ be a function, f is surjective (onto) if for every $b \in B$ there exist $a \in A$ such that $f(a) = b$; we can write in predicate logic formula as

$$\forall y \exists x \ (y = f(x)), \text{ with } x \in A \text{ and } y \in B.$$

If f is surjective, then f is called a surjection.

Remark

Note that $f : A \to B$ is surjective (onto) if **every elements in B has at least one preimage**. We can also say that $f : A \to B$ is surjective if $\operatorname{ran}(f) = \operatorname{Im}(f) = B$.

Example of Surjective Function

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. A function $f : A \to B$ defined as

$$f(a) = 1,\ f(b) = 3,\ f(c) = 1, \text{ and } f(d) = 2$$

is surjective, because for every $y \in B$ there exists $x \in A$ such that $f(x) = y$. For $y = 1$, we have $f(a) = 1$ (and $f(c) = 1$). Also, for $y = 2$, we have $f(d) = 2$. Lastly, for $y = 3$, we have $f(b) = 3$.

Example of Surjective Function

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. A function $f : A \to B$ defined as

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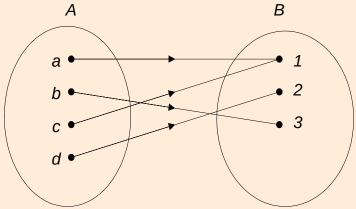
is surjective, because for every $y \in B$ there exists $x \in A$ such that $f(x) = y$. For $y = 1$, we have $f(a) = 1$ (and $f(c) = 1$). Also, for $y = 2$, we have $f(d) = 2$. Lastly, for $y = 3$, we have $f(b) = 3$.



Checking the Surjectivity of a Function

1. To prove that f is surjective, we show that if $y \in B$ then **there is always** an element $x \in A$ such that $f(x) = y$.
   We can also conclude that f is surjective if $\mathrm{ran}(f) = B$.
2. To prove that f is not surjective, we must find $y \in B$ that satisfies $y \neq f(x)$ for all $x \in \mathrm{dom}(f)$.
   We can also conclude that f is not surjective if $\mathrm{ran}(f) \neq B$ (in this case, $\mathrm{ran}(f) \subset B$).

Exercise

Exercise

Check whether the following functions are surjective.

1. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w, x\}$ and $f = \{(1, w), (2, u), (3, v)\}$.
2. $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$ and $f = \{(1, w), (2, u), (3, v)\}$.
3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage.

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage. One of the counterexample is $y =$

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage. One of the counterexample is $y = -1$. There is no $x \in \mathbb{Z}$ satisfies $f(x) = -1$,

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage. One of the counterexample is $y = -1$. There is no $x \in \mathbb{Z}$ satisfies $f(x) = -1$, because this gives $x^2 + 1 = -1 \Rightarrow x^2 = -2$.

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage. One of the counterexample is $y = -1$. There is no $x \in \mathbb{Z}$ satisfies $f(x) = -1$, because this gives $x^2 + 1 = -1 \Rightarrow x^2 = -2$.
4. f is surjective because every $y \in \mathbb{Z}$ has preimage. For every $y \in \mathbb{Z}$ we can choose $x =$

Solution:

1. f is not surjective because $x \in B$ does not have a preimage, or there is no $a \in A$ such that $f(a) = x$.
2. f is surjective because all $b \in B$ have preimage. We have $u = f(2)$, $v = f(3)$, and $w = f(1)$.
3. f is not surjective because not all $y \in \mathbb{Z}$ have preimage. One of the counterexample is $y = -1$. There is no $x \in \mathbb{Z}$ satisfies $f(x) = -1$, because this gives $x^2 + 1 = -1 \Rightarrow x^2 = -2$.
4. f is surjective because every $y \in \mathbb{Z}$ has preimage. For every $y \in \mathbb{Z}$ we can choose $x = y + 1$ such that $f(x) = f(y+1) = (y+1) - 1 = y$.

Contents

Bijective Function

Definition (Bijective function)

Let $f : A \to B$ be a function, f is bijective (one to one correspondence) if f is both injective and surjective. If f is bijective, then f is called a bijection.

Bijective Function Example

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. A function $f : A \to B$ defined as

$$f(a) = 4,\ f(b) = 1,\ f(c) = 3, \text{ and } f(d) = 2$$

is bijective because f is both injective and surjective. Function f is injective because there is no $x, y \in A$ with $f(x) = f(y)$ but $x \neq y$. Also, f is surjective because every $y \in B$ has *preimage*. We have $1 = f(b)$, $2 = f(d)$, $3 = f(c)$, and $4 = f(a)$.

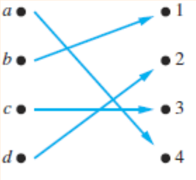
Bijective Function Example

Example

Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. A function $f : A \rightarrow B$ defined as

$$f(a) = 4,\ f(b) = 1,\ f(c) = 3, \text{ and } f(d) = 2$$

is bijective because f is both injective and surjective. Function f is injective because there is no $x, y \in A$ with $f(x) = f(y)$ but $x \neq y$. Also, f is surjective because every $y \in B$ has *preimage*. We have $1 = f(b)$, $2 = f(d)$, $3 = f(c)$, and $4 = f(a)$.



Exercise

Exercise

Check whether these functions are bijective or not.

1. $f: A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$, and $f = \{(1, u), (2, w), (3, v)\}$
2. $f: A \to B$ where $A = \{1, 2, 3\}$ and $B = \{u, v\}$, and $f = \{(1, u), (2, u), (3, v)\}$.
3. $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x - 1$.
4. $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 2x$.

Solution:

1. We have $f(1) = u$, $f(2) = w$, and $f(3) = v$. There is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$, then f is injective. In addition, for every $b \in B$ there exists $a \in A$ such that $b = f(a)$, then f is surjective. Because f injective and surjective, then f is bijective.

Solution:

1. We have $f(1) = u$, $f(2) = w$, and $f(3) = v$. There is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$, then f is injective. In addition, for every $b \in B$ there exists $a \in A$ such that $b = f(a)$, then f is surjective. Because f injective and surjective, then f is bijective.
2. f is not bijective because f is not injective. We have $1 \neq 2$ but $f(1) = f(2) = u$.

Solution:

1. We have $f(1) = u$, $f(2) = w$, and $f(3) = v$. There is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$, then f is injective. In addition, for every $b \in B$ there exists $a \in A$ such that $b = f(a)$, then f is surjective. Because f injective and surjective, then f is bijective.
2. f is not bijective because f is not injective. We have $1 \neq 2$ but $f(1) = f(2) = u$.
3. f is injective because: $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. We also have f is surjective because for every $y \in \mathbb{Z}$ we can choose $x = y + 1$ so that $f(x) = f(y+1) = y + 1 - 1 = y$. Hence, f is bijective.

Solution:

1. We have $f(1) = u$, $f(2) = w$, and $f(3) = v$. There is no $a_1, a_2 \in A$ with $a_1 \neq a_2$ but $f(a_1) = f(a_2)$, then f is injective. In addition, for every $b \in B$ there exists $a \in A$ such that $b = f(a)$, then f is surjective. Because f injective and surjective, then f is bijective.
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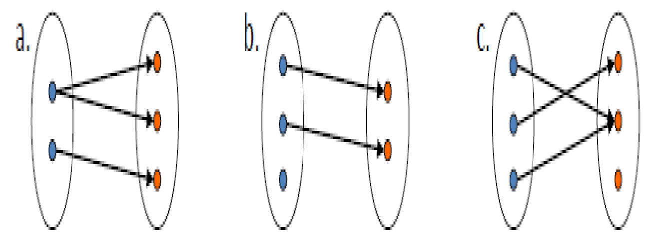
4. f is not bijective because f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 1$. If there is $x \in \mathbb{Z}$ such that $f(x) = 1$, then we have $f(x) = 2x = 1$, so $x = \frac{1}{2} \notin \mathbb{Z}$.

Contents

Exercise

Exercise

From the following relations, which function is injective, surjective, or bijective?



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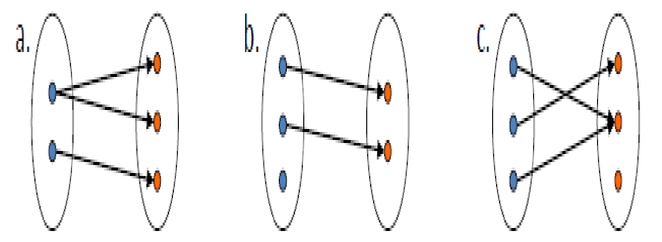

Solution:

- Relation a. is not a function.

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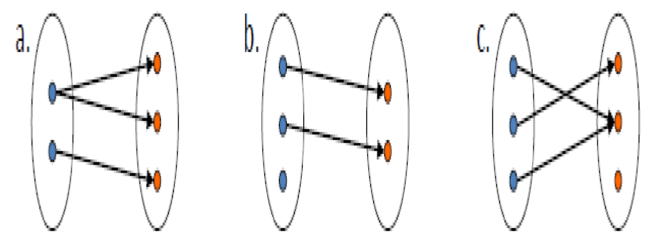

Solution:

- Relation a. is not a function.
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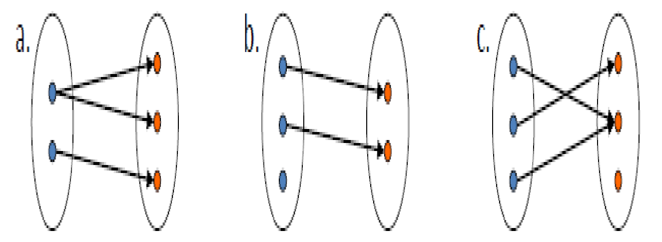

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- Relation a. is not a function.
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- Relation c. is a function but not injective neither surjective.

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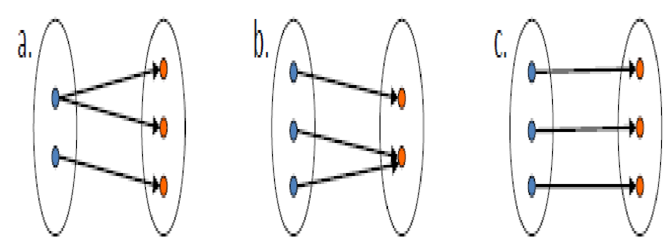
From the following relations, which function is injective, surjective, or bijective?



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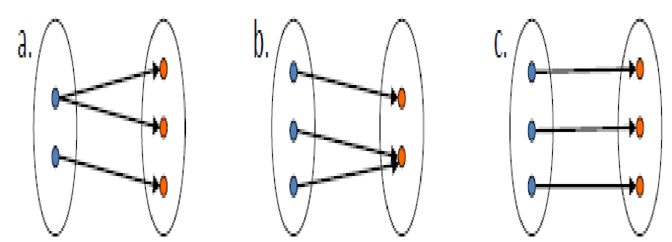

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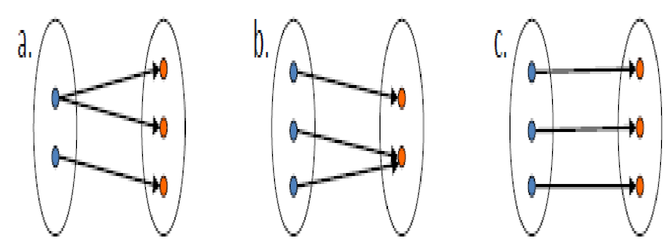

Solution:

- Relation a.is not a function.
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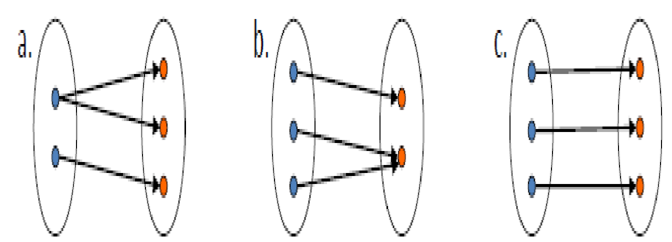
From the following relations, which function is injective, surjective, or bijective?



Solution:

- Relation a.is not a function.
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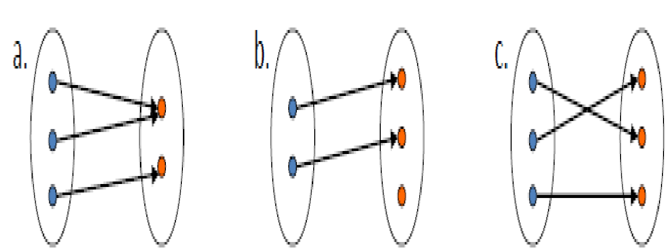

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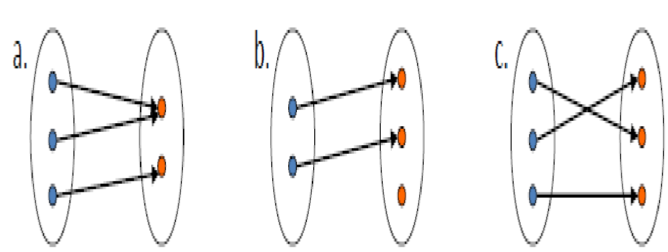

Solution:

- Relation a. is a surjective function, but not injective.

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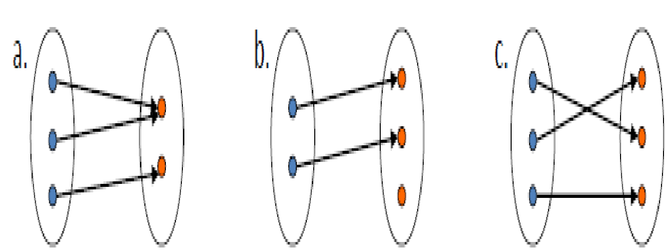

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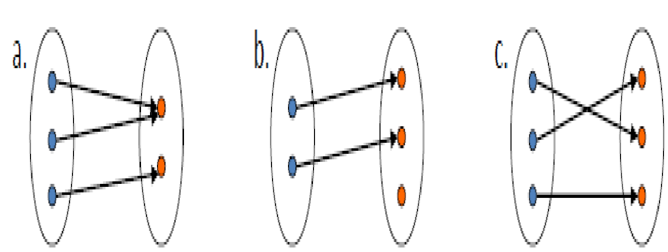

Solution:

- Relation a. is a surjective function, but not injective.
- Relation b. is an injective function, but not surjective.
- Relation c. is a bijective function.

Exercise

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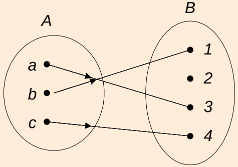
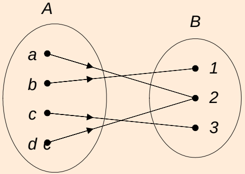
Check whether the relation f that is described by the following arrow diagram is a function. If so, determine whether f is injective, surjective, or bijective.

1. 1. f is:



2. f is:



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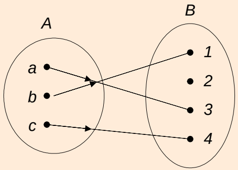

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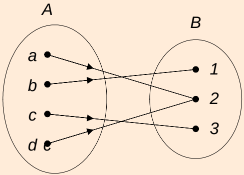

Solution:

1. f is a function from A to B and injective (because image of every $x \in A$ is different) but not surjective because $2 \in B$ does not have a preimage. So, f is not bijective.

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Check whether the relation f that is described by the following arrow diagram is a function. If so, determine whether f is injective, surjective, or bijective.

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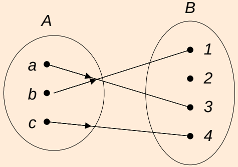

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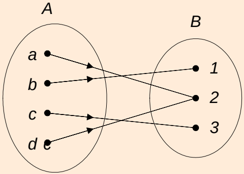


Solution:

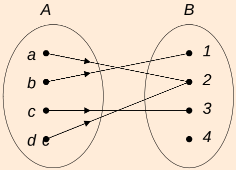
1. f is a function from A to B and injective (because image of every $x \in A$ is different) but not surjective because $2 \in B$ does not have a preimage. So, f is not bijective.
2. f is a function from A to B and surjective (because every $y \in B$ has a preimage) but not injective because $f(a) = f(d) = 2$. So, f is not bijective.

Exercise

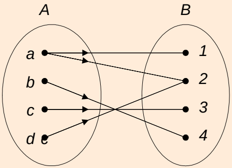
Exercise

Check whether f, represented as arrow diagram, is a function or not. If it is, check whether f is injective, surjective, or bijective.

1. f is:



2. f is:



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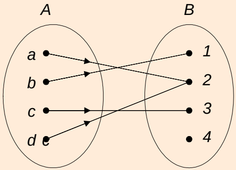

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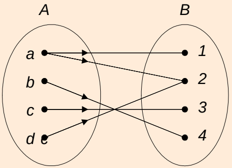

Solution:

1. f is not an injective function because $f(a) = f(d) = 2$. Also, f is not a surjective function because $4 \in B$ does not have a preimage. So, f is not bijective.

Exercise

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Check whether f, represented as arrow diagram, is a function or not. If it is, check whether f is injective, surjective, or bijective.

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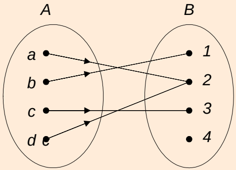

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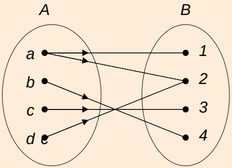

Solution:

1. f is not an injective function because $f(a) = f(d) = 2$. Also, f is not a surjective function because $4 \in B$ does not have a preimage. So, f is not bijective.
2. f is not a function, because $(a, 1) \in f$ and $(a, 2) \in f$. So, f is not injective, surjective, nor bijective.

Exercise

Exercise

Check whether the following functions are injective, surjective, bijective, or none of them.

1. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x + 3$.
2. $f : \mathbb{Z} \to \mathbb{N}_0$ with $f(x) = |x|$, the notation $|x|$ denotes the absolute value of x.
3. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x^2 + 2$.
4. $f : \mathbb{Q} \to \mathbb{Q}$ with $f(x) = 2x + 1$.

Solution:

1. f is injective.

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1. f is injective. Notice that:
   $f(x_1) = f(x_2) \Rightarrow$

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1. f is injective. Notice that:
$f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. But f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 0$.

Solution:

1. f is injective. Notice that:
   $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. But f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ that satisfies $f(x) = 0$, then $f(x) = 2x + 3 = 0$,

Solution:

1. f is injective. Notice that:
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2. f is not injective because $f(-1) = f(1) = |-1| = |1| = 1$.

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Solution:

1. f is injective. Notice that:
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1. f is injective. Notice that: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. But f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ that satisfies $f(x) = 0$, then $f(x) = 2x + 3 = 0$, so $x = -\frac{3}{2} \notin \mathbb{Z}$. Then f is not bijective.
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Solution:

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   $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. But f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ that satisfies $f(x) = 0$, then $f(x) = 2x + 3 = 0$, so $x = -\frac{3}{2} \notin \mathbb{Z}$. Then f is not bijective.
2. f is not injective because $f(-1) = f(1) = |-1| = |1| = 1$. The function f is surjective because for every $y \in \mathbb{N}_0$ there is $x = y \in \mathbb{Z}$ such that $f(x) = |x| = x = y$. Hence, f is not bijective.
3. f is not injective because $f(-1) = f(1) = 3$. Moreover f is not surjective because there is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ satisfies $f(x) = 0$, then $f(x) = x^2 + 2 = 0 \Rightarrow x^2 = -2$, which is not possible for all $x \in \mathbb{Z}$.
4. f is injective because
   $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. The function f is surjective because for every $y \in \mathbb{Q}$, we can choose $x = \frac{y-1}{2} \in \mathbb{Q}$. So, $f(x) = f\left(\frac{y-1}{2}\right) =$

Solution:

1. f is injective. Notice that: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. But f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ that satisfies $f(x) = 0$, then $f(x) = 2x + 3 = 0$, so $x = -\frac{3}{2} \notin \mathbb{Z}$. Then f is not bijective.
2. f is not injective because $f(-1) = f(1) = |-1| = |1| = 1$. The function f is surjective because for every $y \in \mathbb{N}_0$ there is $x = y \in \mathbb{Z}$ such that $f(x) = |x| = x = y$. Hence, f is not bijective.
3. f is not injective because $f(-1) = f(1) = 3$. Moreover f is not surjective because there is no $x \in \mathbb{Z}$ such that $f(x) = 0$. If there is $x \in \mathbb{Z}$ satisfies $f(x) = 0$, then $f(x) = x^2 + 2 = 0 \Rightarrow x^2 = -2$, which is not possible for all $x \in \mathbb{Z}$.
4. f is injective because $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. The function f is surjective because for every $y \in \mathbb{Q}$, we can choose $x = \frac{y-1}{2} \in \mathbb{Q}$. So, $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1 = y - 1 + 1 = y$. Therefore, f is bijective.

Challenging Problem

Exercise

Check whether these functions is injective, surjective, bijective, or none of them.

1. $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{1\}$ with $f(x) = \frac{x}{x-1}$.
2. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} 2x+1, & \text{if } x \leq 1 \\ 4x+3, & \text{if } x > 1. \end{cases}$.
3. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} 2x+1, & \text{if } x > 1 \\ 4x+3, & \text{if } x \leq 1. \end{cases}$.

Contents

Function Composition

Definition

Let A, B, C be three sets, $f : A \to B$ and $g : B \to C$. Function composition of g and f is function $g \circ f : A \to C$ defined as

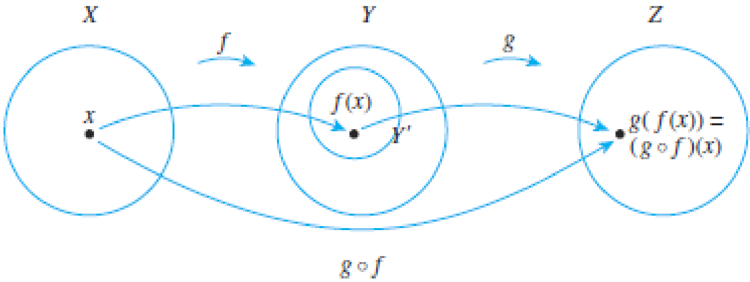
$$(g \circ f)(x) = g(f(x))$$

for every $x \in \operatorname{dom}(f)$.

In order for $g \circ f$ to be defined, it should be $\operatorname{ran}(f) \subseteq \operatorname{dom}(g)$.

Illustration of Function Composition

Let $f : X \to Y$ and $g : Y \to Z$ be two functions with $\mathrm{ran}\,(f) = Y' \subseteq Y$, such that $\mathrm{ran}\,(f) \subseteq \mathrm{dom}\,(g)$. Function composition $g \circ f$ can be illustrated below.



We have $(g \circ f)\,(x) = g\,(f\,(x))$ for every $x \in X$.

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be two functions defined as:
$f = \{(1, c), (2, b), (3, a)\}$ and
$g = \{(a, y), (b, y), (c, z), (d, z), (e, z)\}$.
We have the following illustration:

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be two functions defined as:
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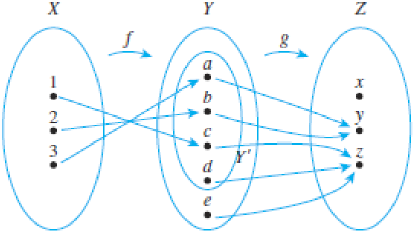

We can see that:
$(g \circ f)(1) =$

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions defined as:
$f = \{(1, c), (2, b), (3, a)\}$ and
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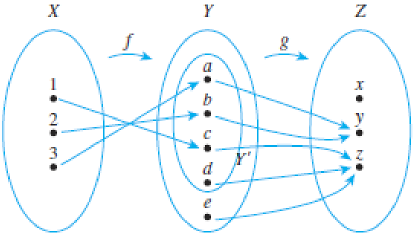

We can see that:
$(g \circ f)(1) = g(f(1)) = g(c) = z$,
$(g \circ f)(2) =$

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be two functions defined as:
$f = \{(1, c), (2, b), (3, a)\}$ and
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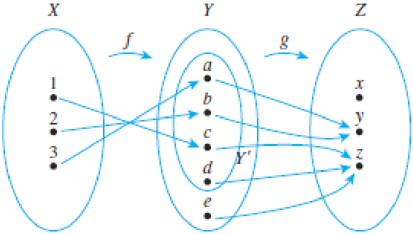
We have the following illustration:



We can see that:
$(g \circ f)(1) = g(f(1)) = g(c) = z$,
$(g \circ f)(2) = g(f(2)) = g(b) = y$, and
$(g \circ f)(3) =$

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be two functions defined as:
$f = \{(1, c), (2, b), (3, a)\}$ and
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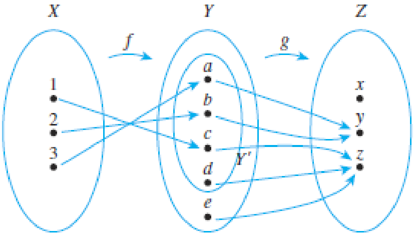

We can see that:
$(g \circ f)(1) = g(f(1)) = g(c) = z$,
$(g \circ f)(2) = g(f(2)) = g(b) = y$, and
$(g \circ f)(3) = g(f(3)) = g(a) = y$.
Then $g \circ f =$

Function Composition Example

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Let $f : X \to Y$ and $g : Y \to Z$ be two functions defined as:
$f = \{(1, c), (2, b), (3, a)\}$ and
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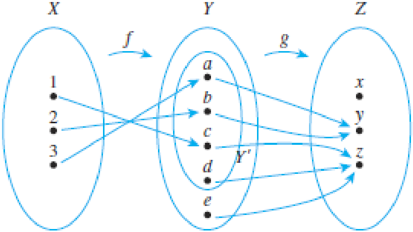
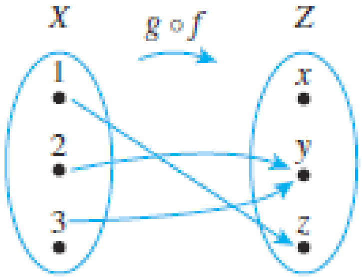
We have the following illustration:



We can see that:
$(g \circ f)(1) = g(f(1)) = g(c) = z$,
$(g \circ f)(2) = g(f(2)) = g(b) = y$, and
$(g \circ f)(3) = g(f(3)) = g(a) = y$.
Then $g \circ f = \{(1, z), (2, y), (3, y)\}$.

Note that $g \circ f$ is a function from X to Z with $\operatorname{ran}(g \circ f) = \operatorname{Im}(g \circ f) = \{y, z\}$.

Note that $g \circ f$ is a function from X to Z with $\operatorname{ran}(g \circ f) = \operatorname{Im}(g \circ f) = \{y, z\}$.



Exercise

Exercise

If possible, determine the composition of the following functions.

1. $f : \{a, b, c\} \to \{a, b, c\}$ with $f(a) = b$, $f(b) = c$, $f(c) = a$ and $g : \{a, b, c\} \to \{1, 2, 3\}$ with $g(a) = 1$, $g(b) = 2$, $g(c) = 3$. Find $f \circ f$, $f \circ f \circ f$, $g \circ f$, and $f \circ g$.
2. $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x - 1$ and $g(x) = x^2$, find the formula for $(f \circ g)(x)$ and $(g \circ f)(x)$.
3. $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x$ and $g(x) = 1$, find the formula for $(f \circ g)(x)$ and $(g \circ f)(x)$.
4. $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 1$ and $g(x) = 2$, find the formula for $(f \circ g)(x)$ and $(g \circ f)(x)$.
5. $f, g : \mathbb{Q} \to \mathbb{Q}$ with $f(x) = 2x - 1$ and $g(x) = \frac{x+1}{2}$, find the formula for $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution:

1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$.

Solution:

1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$.

Solution:

1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$. The function $g \circ f$ is a function defined as: $(g \circ f)(a) = 2$, $(g \circ f)(b) = 3$, $(g \circ f)(c) = 1$.

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1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$. The function $g \circ f$ is a function defined as: $(g \circ f)(a) = 2$, $(g \circ f)(b) = 3$, $(g \circ f)(c) = 1$. Lastly, because $\operatorname{dom}(f) = \{a, b, c\}$ and $\operatorname{ran}(g) = \{1, 2, 3\}$, then $f \circ g$ is not defined.
2. We have
   $(f \circ g)(x) = f(g(x)) =$

Solution:

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   $(f \circ g)(x) = f(g(x)) = g(x) - 1 = x^2 - 1.$
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1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$. The function $g \circ f$ is a function defined as: $(g \circ f)(a) = 2$, $(g \circ f)(b) = 3$, $(g \circ f)(c) = 1$. Lastly, because $\operatorname{dom}(f) = \{a, b, c\}$ and $\operatorname{ran}(g) = \{1, 2, 3\}$, then $f \circ g$ is not defined.
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   $(f \circ g)(x) = f(g(x)) = 1.$
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Solution:

1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$. The function $g \circ f$ is a function defined as: $(g \circ f)(a) = 2$, $(g \circ f)(b) = 3$, $(g \circ f)(c) = 1$. Lastly, because $\mathrm{dom}(f) = \{a, b, c\}$ and $\mathrm{ran}(g) = \{1, 2, 3\}$, then $f \circ g$ is not defined.
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   $(g \circ f)(x) = g(f(x)) =$

Solution:

1. We have $f \circ f$ is a function defined as: $(f \circ f)(a) = c$, $(f \circ f)(b) = a$, $(f \circ f)(c) = b$. Next, $f \circ f \circ f$ is a function defined as: $(f \circ f \circ f)(a) = a$, $(f \circ f \circ f)(b) = b$, $(f \circ f \circ f)(c) = c$. The function $g \circ f$ is a function defined as: $(g \circ f)(a) = 2$, $(g \circ f)(b) = 3$, $(g \circ f)(c) = 1$. Lastly, because $\operatorname{dom}(f) = \{a, b, c\}$ and $\operatorname{ran}(g) = \{1, 2, 3\}$, then $f \circ g$ is not defined.
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   $(f \circ g)(x) = f(g(x)) = 2g(x) - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$.
   $(g \circ f)(x) = g(f(x)) = \frac{f(x)+1}{2} = \frac{(2x-1)+1}{2} = \frac{2x}{2} = x$.

Contents

Inverse Function

Definition

Let $f : A \rightarrow B$ be a bijective function. Inverse function of f is function $f^{-1} : B \rightarrow A$ such that

$$\begin{aligned} \left(f^{-1} \circ f\right)(a) &= f^{-1}(f(a)) = a, \\ \left(f \circ f^{-1}\right)(b) &= f\left(f^{-1}(b)\right) = b, \end{aligned}$$

for every $a \in A$ and $b \in B$. If f has inverse, then f is *invertible*.

REMEMBER: the requirement for a function $f : A \rightarrow B$ to have an inverse is f should have **bijective** property (as a one-to-one correspondence). If $f : A \rightarrow B$ is not bijective, then f^{-1} is not defined.

Inverse Function Example

Example

Let $f : A \rightarrow B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$.

Inverse Function Example

Example

Let $f : A \rightarrow B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$. Inverse function of f is f^{-1} with the properties $\left(f \circ f^{-1}\right)(b) = b$ and $\left(f^{-1} \circ f\right)(a) = a$ for all $a \in A$ and $b \in B$. We have

$$f^{-1}(u) =$$

Inverse Function Example

Example

Let $f : A \to B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$. Inverse function of f is f^{-1} with the properties $\left(f \circ f^{-1}\right)(b) = b$ and $\left(f^{-1} \circ f\right)(a) = a$ for all $a \in A$ and $b \in B$. We have

$$f^{-1}(u) = 2,\ f^{-1}(v) =$$

Inverse Function Example

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Let $f : A \rightarrow B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$. Inverse function of f is f^{-1} with the properties $\left(f \circ f^{-1}\right)(b) = b$ and $\left(f^{-1} \circ f\right)(a) = a$ for all $a \in A$ and $b \in B$. We have

$$f^{-1}(u) = 2,\ f^{-1}(v) = 3, \text{ and } f^{-1}(w) =$$

Inverse Function Example

Example

Let $f : A \rightarrow B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$. Inverse function of f is f^{-1} with the properties $\left(f \circ f^{-1}\right)(b) = b$ and $\left(f^{-1} \circ f\right)(a) = a$ for all $a \in A$ and $b \in B$. We have

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$$f^{-1}(u) = 2,\ f^{-1}(v) = 3, \text{ and } f^{-1}(w) = 1.$$

Notice that

$$\begin{aligned}\left(f \circ f^{-1}\right)(u) &= f\left(f^{-1}(u)\right) = f(2) = u, \\ \left(f \circ f^{-1}\right)(v) &= \end{aligned}$$

Inverse Function Example

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using the similar idea, we can also prove that

Inverse Function Example

Example

Let $f : A \rightarrow B$ with $A = \{1, 2, 3\}$ and $B = \{u, v, w\}$,and $f = \{(1, w), (2, u), (3, v)\}$. Function f has bijective properties (as one-to-one correspondence). We have $f(1) = w$, $f(2) = u$, and $f(3) = v$. Inverse function of f is f^{-1} with the properties $\left(f \circ f^{-1}\right)(b) = b$ and $\left(f^{-1} \circ f\right)(a) = a$ for all $a \in A$ and $b \in B$. We have

$$f^{-1}(u) = 2,\ f^{-1}(v) = 3, \text{ and } f^{-1}(w) = 1.$$

Notice that

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\left(f \circ f^{-1}\right)(u) &= f\left(f^{-1}(u)\right) = f(2) = u, \\
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\left(f \circ f^{-1}\right)(w) &= f\left(f^{-1}(w)\right) = f(1) = w,
\end{aligned}$$

using the similar idea, we can also prove that $\left(f^{-1} \circ f\right)(1) = 1$, $\left(f^{-1} \circ f\right)(2) = 2$, and $\left(f^{-1} \circ f\right)(3) = 3$.

Exercise

Exercise

Find (if exists) the inverse of these following functions:

1. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x - 1$.
2. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x^2 + 1$.
3. $f : \mathbb{Q}^+ \to \mathbb{Q}$ with $f(x) = \frac{x-1}{x}$.
4. $f : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x$.

Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x =$

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Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that
   $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and
   $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.

Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.
2. f is not bijective because f is not injective neither surjective. We have $f(1) = f(-1) = 2$ and there is no $x \in \mathbb{Z}$ such that $f(x) = 0$. Thus, f has no inverse.

Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.
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Solution:

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Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.
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Solution:

1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.
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1. f is bijective because f is injective and surjective (prove it!). If $f(x) = x - 1 = y$, then $x = y + 1$, so $f^{-1}(y) = y + 1$, then we have $f^{-1}(x) = x + 1$. Notice that $\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = f(x+1) = (x+1) - 1 = x$ and $\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = f^{-1}(x-1) = (x-1) + 1 = x$.
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4. f is not bijective because f is not surjective. There is no $x \in \mathbb{Z}$ such that $f(x) = 1$.

Contents

Floor Function and Ceiling Function

Definition

Floor function maps the real number x to the greatest integer smaller than or equal to x. *Floor function* is denoted by $\lfloor \cdots \rfloor$. Formally, for every $x \in \mathbb{R}$, $\lfloor x \rfloor = n$ where $n \leq x < n + 1$.

Definition

Ceiling function maps the real number x to the smallest integer greater than or equal to x. *Ceiling function* is denoted by $\lceil \cdots \rceil$. Formally, for every $x \in \mathbb{R}$, $\lceil x \rceil = m$ where $m - 1 < x \leq m$.

Intuitively: $\lfloor x \rfloor$ rounds x "down", while $\lceil x \rceil$ rounds x "up".

Examples (Floor and Ceiling)

Example

We have

1. $\lfloor 3.5 \rfloor =$

Examples (Floor and Ceiling)

Example

We have

1. $\lfloor 3.5 \rfloor = 3$ and $\lceil 3.5 \rceil =$

Examples (Floor and Ceiling)

Example

We have

1. $\lfloor 3.5 \rfloor = 3$ and $\lceil 3.5 \rceil = 4$.
2. $\lfloor 0.7 \rfloor =$

Examples (Floor and Ceiling)

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Examples (Floor and Ceiling)

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1. $\lfloor 3.5 \rfloor = 3$ and $\lceil 3.5 \rceil = 4$.
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Examples (Floor and Ceiling)

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Examples (Floor and Ceiling)

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Examples (Floor and Ceiling)

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7. $\lfloor -1.3 \rfloor =$

Examples (Floor and Ceiling)

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Examples (Floor and Ceiling)

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8. $\lfloor -4 \rfloor =$

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Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\left\lfloor \sqrt{2} \right\rfloor$ and $\left\lceil \sqrt{2} \right\rceil$
8) $\left\lfloor -\sqrt{2} \right\rfloor$ and $\left\lceil -\sqrt{2} \right\rceil$
9) $\left\lfloor -3\sqrt{2} \right\rfloor$ and $\left\lceil -3\sqrt{2} \right\rceil$
10) $\left\lfloor 2\sqrt{3} \right\rfloor$ and $\left\lceil 2\sqrt{3} \right\rceil$

Solution: **1)**

Exercise

Exercise

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2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
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Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)**

Exercise

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Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
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Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)**

Exercise

Exercise

Find:

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10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\left\lfloor \sqrt{2} \right\rfloor$ and $\left\lceil \sqrt{2} \right\rceil$
8) $\left\lfloor -\sqrt{2} \right\rfloor$ and $\left\lceil -\sqrt{2} \right\rceil$
9) $\left\lfloor -3\sqrt{2} \right\rfloor$ and $\left\lceil -3\sqrt{2} \right\rceil$
10) $\left\lfloor 2\sqrt{3} \right\rfloor$ and $\left\lceil 2\sqrt{3} \right\rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\lfloor \sqrt{2} \rfloor$ and $\lceil \sqrt{2} \rceil$
8) $\lfloor -\sqrt{2} \rfloor$ and $\lceil -\sqrt{2} \rceil$
9) $\lfloor -3\sqrt{2} \rfloor$ and $\lceil -3\sqrt{2} \rceil$
10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\lfloor \sqrt{2} \rfloor$ and $\lceil \sqrt{2} \rceil$
8) $\lfloor -\sqrt{2} \rfloor$ and $\lceil -\sqrt{2} \rceil$
9) $\lfloor -3\sqrt{2} \rfloor$ and $\lceil -3\sqrt{2} \rceil$
10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)** $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$, **7)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\left\lfloor \sqrt{2} \right\rfloor$ and $\left\lceil \sqrt{2} \right\rceil$
8) $\left\lfloor -\sqrt{2} \right\rfloor$ and $\left\lceil -\sqrt{2} \right\rceil$
9) $\left\lfloor -3\sqrt{2} \right\rfloor$ and $\left\lceil -3\sqrt{2} \right\rceil$
10) $\left\lfloor 2\sqrt{3} \right\rfloor$ and $\left\lceil 2\sqrt{3} \right\rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)** $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$, **7)** $\left\lfloor \sqrt{2} \right\rfloor = 1$ and $\left\lceil \sqrt{2} \right\rceil = 2$, **8)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\lfloor \sqrt{2} \rfloor$ and $\lceil \sqrt{2} \rceil$
8) $\lfloor -\sqrt{2} \rfloor$ and $\lceil -\sqrt{2} \rceil$
9) $\lfloor -3\sqrt{2} \rfloor$ and $\lceil -3\sqrt{2} \rceil$
10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)** $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$, **7)** $\lfloor \sqrt{2} \rfloor = 1$ and $\lceil \sqrt{2} \rceil = 2$, **8)** $\lfloor -\sqrt{2} \rfloor = -2$ and $\lceil -\sqrt{2} \rceil = -1$, **9)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\lfloor \sqrt{2} \rfloor$ and $\lceil \sqrt{2} \rceil$
8) $\lfloor -\sqrt{2} \rfloor$ and $\lceil -\sqrt{2} \rceil$
9) $\lfloor -3\sqrt{2} \rfloor$ and $\lceil -3\sqrt{2} \rceil$
10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)** $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$, **7)** $\lfloor \sqrt{2} \rfloor = 1$ and $\lceil \sqrt{2} \rceil = 2$, **8)** $\lfloor -\sqrt{2} \rfloor = -2$ and $\lceil -\sqrt{2} \rceil = -1$, **9)** $\lfloor -3\sqrt{2} \rfloor = -5$ and $\lceil -3\sqrt{2} \rceil = -4$, **10)**

Exercise

Exercise

Find:

1) $\lfloor 2.8 \rfloor$ and $\lceil 2.8 \rceil$
2) $\lfloor 3.1 \rfloor$ and $\lceil 3.1 \rceil$
3) $\lfloor -1.4 \rfloor$ and $\lceil -1.4 \rceil$
4) $\lfloor -2.7 \rfloor$ and $\lceil -2.7 \rceil$
5) $\lfloor \pi \rfloor$ and $\lceil \pi \rceil$
6) $\lfloor -\pi \rfloor$ and $\lceil -\pi \rceil$
7) $\lfloor \sqrt{2} \rfloor$ and $\lceil \sqrt{2} \rceil$
8) $\lfloor -\sqrt{2} \rfloor$ and $\lceil -\sqrt{2} \rceil$
9) $\lfloor -3\sqrt{2} \rfloor$ and $\lceil -3\sqrt{2} \rceil$
10) $\lfloor 2\sqrt{3} \rfloor$ and $\lceil 2\sqrt{3} \rceil$

Solution: **1)** $\lfloor 2.8 \rfloor = 2$ and $\lceil 2.8 \rceil = 3$, **2)** $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$, **3)** $\lfloor -1.4 \rfloor = -2$ and $\lceil -1.4 \rceil = -1$, **4)** $\lfloor -2.7 \rfloor = -3$ and $\lceil -2.7 \rceil = -2$, **5)** $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$, **6)** $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$, **7)** $\lfloor \sqrt{2} \rfloor = 1$ and $\lceil \sqrt{2} \rceil = 2$, **8)** $\lfloor -\sqrt{2} \rfloor = -2$ and $\lceil -\sqrt{2} \rceil = -1$, **9)** $\lfloor -3\sqrt{2} \rfloor = -5$ and $\lceil -3\sqrt{2} \rceil = -4$, **10)** $\lfloor 2\sqrt{3} \rfloor = 3$ and $\lceil 2\sqrt{3} \rceil = 4$.

Modulo (mod) and Divisor (div Functions)

Theorem

Let $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, then there is $q \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $0 \leq r < m$ such that

$$a = mq + r,$$

Integers q and r are **unique** for every a and m. Furthermore:

1. the value of q is called as a *quotient* of a divided by m and is denoted as $a \operatorname{div} m$;
2. the value of r is called as a *remainder* of a divided by m and is denoted as $a \bmod m$ (the value of the remainder is never negative).

mod and div will be discussed further in elementary number theory.

Example

Example

We have

1. $25 \bmod 7 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4\,(6) + 3$.
6. $-27 \bmod 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4\,(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 = -7$, because $-27 = 4\,(-7) + 1$.
7. $37 \bmod 6 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4\,(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 = -7$, because $-27 = 4\,(-7) + 1$.
7. $37 \bmod 6 = 1$ and $37 \operatorname{div} 6 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4\,(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 = -7$, because $-27 = 4\,(-7) + 1$.
7. $37 \bmod 6 = 1$ and $37 \operatorname{div} 6 = 6$, because $37 = 6\,(6) + 1$.
8. $-37 \bmod 6 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7\,(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4\,(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45\,(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5\,(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4\,(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 = -7$, because $-27 = 4\,(-7) + 1$.
7. $37 \bmod 6 = 1$ and $37 \operatorname{div} 6 = 6$, because $37 = 6\,(6) + 1$.
8. $-37 \bmod 6 = 5$ and $-37 \operatorname{div} 6 =$

Example

Example

We have

1. $25 \bmod 7 = 4$ and $25 \operatorname{div} 7 = 3$, because $25 = 7(3) + 4$.
2. $16 \bmod 4 = 0$ and $16 \operatorname{div} 4 = 4$, because $16 = 4(4) + 0$.
3. $4512 \bmod 45 = 12$ and $4512 \operatorname{div} 45 = 100$, because $4512 = 45(100) + 12$.
4. $0 \bmod 5 = 0$ and $0 \operatorname{div} 5 = 0$, because $0 = 5(0) + 0$.
5. $27 \bmod 4 = 3$ and $27 \operatorname{div} 4 = 6$, because $27 = 4(6) + 3$.
6. $-27 \bmod 4 = 1$ and $-27 \operatorname{div} 4 = -7$, because $-27 = 4(-7) + 1$.
7. $37 \bmod 6 = 1$ and $37 \operatorname{div} 6 = 6$, because $37 = 6(6) + 1$.
8. $-37 \bmod 6 = 5$ and $-37 \operatorname{div} 6 = -7$, because $-37 = 6(-7) + 5$.

Factorial Function

Definition

A factorial function is a function from $\mathbb{N}_0$ to $\mathbb{N}$ defined as

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n \times (n-1) \times \cdots \times 2 \times 1, & \text{if } n > 0 \end{cases}$$

For example, we have $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$.

Exponential Function

Definition

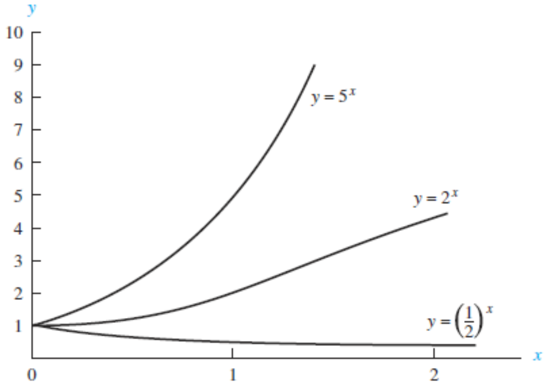
Let $a \in \mathbb{R}$ and $a \neq 0$. An exponential function is defined as:

1. For $n \in \mathbb{N}_0$, then

$$a^n = \begin{cases} 1, & \text{if } n = 0 \\ \underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}, & \text{if } n > 0 \end{cases}$$

2. For $n \in \mathbb{Z}$, if $n = -m < 0$, then $a^n = a^{-m} = \frac{1}{a^m}$,
3. For $q \in \mathbb{Q}$, if $q = \frac{m}{n}$ with $m, n \in \mathbb{Z}$ and $n \neq 0$, then $a^q = a^{\frac{m}{n}} = \sqrt[n]{a^m}$,
4. For $x \in \mathbb{R}$, if x is irrational, then a^x defined as $a^x = e^{x \ln a}$, where $\ln a$ is natural logarithm of a.

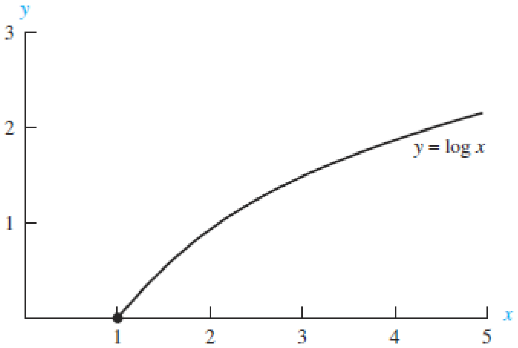
Example of Exponential Function



Logarithmic Function

Logarithmic Function

From an expression $y = a^x$, we have $x = {}^a\log y = \log_a y$. The function $f(x) = \log_a x$ with $a > 0$ is a logarithmic function with base a.



Recursive Function

Recursive Function

A function f is called a **recursive function** if its definition is referred to f itself. A recursive function consists of a *base case* (or *base cases*) and a *recursive case* (or *recursive cases*).

Example

The factorial function can be defined recursively:

$$n! = \begin{cases} 1, & \text{if } n = 0 \\ n \times (n-1)!, & \text{if } n > 0. \end{cases}$$

We have $0! = 0$, $1! = 1 \cdot 0! = 1$, $2! = 2 \cdot 1! = 2$, and so forth. Case $n! = 1$ if $n = 0$ is called as a *base case*, while case $n! = n \times (n-1)!$ is called as a *recursive case*.

Recursive Function and Recursive Algorithm

A recursive function can be defined using a particular formula or using a program in a particular programming language.

Example

The function $f : \mathbb{N} \to \mathbb{N}$ defined recursively as:

$$f(n) = \begin{cases} 1, & n = 1 \\ 2, & n = 2 \\ f(n-1) + f(n-2), & n \geq 3 \end{cases}$$

can also be defined by using Python:

```
def f(n):
    if n == 1: return 1
    if n == 2: return 2
    else: return f(n-1) + f(n-2)
```

Exercise

Find $f(5)$, $f(6)$, and $f(7)$.

Contents

Challenging Problems

Ackermann Function

Ackermann function is an important function in theoretical computer science due to its prevalence in recursive algorithm concerning sets. One type of this function is $A : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ which is defined as:

$$A(m,n) = \begin{cases} 2n, & \text{if } m = 0 \\ 0, & \text{if } m \geq 1 \text{ and } n = 0 \\ 2 & \text{if } m \geq 1 \text{ and } n = 1 \\ A(m-1, A(m, n-1)) & \text{if } m \geq 1 \text{ and } n \geq 2 \end{cases}.$$

Determine the value of $A(2,2)$, $A(2,3)$, and $A(3,3)$.

Nearest Power of 2

Computer usually processes numbers in their bit expressions (a base 2 number). For instance:

$$2 := \mathbf{10},\ 4 := \mathbf{100},\ 6 := \mathbf{110},\ 7 := \mathbf{111},\ 10 := \mathbf{1010}$$

In order to represent a positive integer n in its bit expression, we need to know its bit length (the number of digits required) Suppose the minimum bit length for representing a number n is $\ell(n)$. Thus, we have

$$\ell(2) = 2,\ \ell(4) = 3,\ \ell(6) = 3,\ \ell(7) = 3,\ \ell(10) = 4.$$

Basically, $\ell(n)$ is the least integer k such that $n \leq 2^k$. Give a formal-mathematical definition of $\ell(n)$.

Piecewise Function

Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} 4x+3, & \text{if } x \leq 1 \\ 2x+1, & \text{if } x > 1. \end{cases}$. Check whether f is injective, surjective, bijective, or none of them.