Relation: Properties and Composition

Discrete Mathematics – Second Term 2022-2023

MZI

School of Computing
Telkom University

SoC Tel-U

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, K. H. Rosen (primary).
2. *Discrete Mathematics with Applications* , 5th Edition, 2018, S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, B. H. Widjaja.
5. Slide for Matematika Diskrit. Telkom University, B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to <pleasedontspam>@telkomuniversity.ac.id.

Contents

Contents

Complement, $\vee$, and $\wedge$ on $0, 1$

Definition

If $a \in \{0, 1\}$, then $\bar{a} = \neg a = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{if } a = 1. \end{cases}$

Definition

Operation $\wedge$ and $\vee$ on set $\{0, 1\}$ defined as: for every $a, b \in \{0, 1\}$

$$a \vee b = \begin{cases} 1, & \text{if } a = 1 \text{ or } b = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$a \wedge b = \begin{cases} 1, & \text{if } a = b = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Complement, $\vee$, and $\wedge$ on $0-1$ Matrices

Definition

Let $\mathbf{A} = [a_{ij}]$ be a $0-1$ matrix with m rows and n columns, then the complement matrix $\overline{\mathbf{A}}$ (or $\neg\mathbf{A}$) is an $m \times n$ matrix whose i-th row and j-th row entry is $\overline{\mathbf{A}}\,[i,j] = [\bar{a}_{ij}]$,

Definition

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two $0-1$ matrices with m rows and n columns, then the conjunction $\mathbf{A} \wedge \mathbf{B}$ and disjunction $\mathbf{A} \vee \mathbf{B}$ respectively $m \times n$ matrices whose i-th row and j-th row entry is

$$\begin{aligned}(\mathbf{A} \wedge \mathbf{B})\,[i,j] &= \mathbf{A}\,[i,j] \wedge \mathbf{B}\,[i,j] = [a_{ij} \wedge b_{ij}]\,, \\ (\mathbf{A} \vee \mathbf{B})\,[i,j] &= \mathbf{A}\,[i,j] \vee \mathbf{B}\,[i,j] = [a_{ij} \vee b_{ij}]\,.\end{aligned}$$

Representation Matrices for R^{-1}, $\bar{R}$, $R_1 \cup R_2$, and $R_1 \cap R_2$

Theorem

Let R be a relation from A to B (both are finite sets). If $\mathbf{M}_R$ is a representation matrix for R, then representation matrix for relation R^{-1} which is a relation from B to A, noted as $\mathbf{M}_{R^{-1}}$, satisfies $\mathbf{M}_{R^{-1}} = (\mathbf{M}_R)^T$.

Theorem

Let R be a relation from A to B (both are finite sets). If $\mathbf{M}_R$ is a representation matrix for R, then representation matrix for relation $\bar{R}$, noted as $\mathbf{M}_{\bar{R}}$, satisfies $\mathbf{M}_{\bar{R}} = \overline{\mathbf{M}}_R$.

Theorem

Let R_1 and R_2 be two relations from A to B. If $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ are representation matrices of relation R_1 and R_2, respectively, then the representation matrix for $R_1 \cup R_2$ and $R_1 \cap R_2$ are

$$\begin{aligned} \mathbf{M}_{R_1 \cup R_2} &= \mathbf{M}_{R_1} \vee \mathbf{M}_{R_1} \text{ and} \\ \mathbf{M}_{R_1 \cap R_2} &= \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}. \end{aligned}$$

Exercise

Let R_1 and R_2 be two relations on $A = \{a, b, c\}$ and their representation matrices are:

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find representation matrix for $\bar{R}_1$, R_1^{-1}, $R_1 \cup R_2$, and $R_1 \cap R_2$

Solution: $\mathbf{M}_{\bar{R}_1} =$

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Contents

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Reflexive and Irreflexive Relations

Definition

Let R be a binary relation on A, relation R is:

1. reflexive, if for every $a \in A$ we have $(a, a) \in R$, or aRa
   (R is reflexive if: $\forall a\,(aRa)$ or $\forall a\,((a, a) \in R)$).
2. irreflexive, if for every $a \in A$ we have $(a, a) \notin R$, or $a \not R a$.
   (R irreflexive if: $\forall a\,(a \not R a)$ or $\forall a\,((a, a) \notin R)$ or $\forall a\,(\neg\,(a, a) \in R)$).

Question:

Reflexive and Irreflexive Relations

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Question: Does irreflexive mean not reflexive?

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, and R_3 be relations on A defined as:

1. $R_1 = \{(1,1), (1,3), (2,1), (2,2), (3,3), (4,2), (4,3), (4,4)\}$.
2. $R_2 = \{(1,2), (2,3), (3,4), (4,1)\}$.
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We have:

1. Relation R_1 is reflexive because $(1,1) \in R_1$, $(2,2) \in R_1$, $(3,3) \in R_1$, $(4,4) \in R_1$.

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We have:

1. Relation R_1 is reflexive because $(1,1) \in R_1$, $(2,2) \in R_1$, $(3,3) \in R_1$, $(4,4) \in R_1$. Relation R_1 **is not** irreflexive because $(1,1) \notin R_1$ is not true.
2. Relation R_2 **is not** reflexive because $(1,1) \notin R_2$.

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, and R_3 be relations on A defined as:

1. $R_1 = \{(1,1), (1,3), (2,1), (2,2), (3,3), (4,2), (4,3), (4,4)\}$.
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We have:

1. Relation R_1 is reflexive because $(1,1) \in R_1$, $(2,2) \in R_1$, $(3,3) \in R_1$, $(4,4) \in R_1$. Relation R_1 **is not** irreflexive because $(1,1) \notin R_1$ is not true.
2. Relation R_2 **is not** reflexive because $(1,1) \notin R_2$. Relation R_2 is irreflexive because $(1,1) \notin R_2$, $(2,2) \notin R_2$, $(3,3) \notin R_2$, and $(4,4) \notin R_2$.

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Let $A = \{1, 2, 3, 4\}$, R_1, R_2, and R_3 be relations on A defined as:

1. $R_1 = \{(1,1), (1,3), (2,1), (2,2), (3,3), (4,2), (4,3), (4,4)\}$.
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2. Relation R_2 **is not** reflexive because $(1,1) \notin R_2$. Relation R_2 is irreflexive because $(1,1) \notin R_2$, $(2,2) \notin R_2$, $(3,3) \notin R_2$, and $(4,4) \notin R_2$.
3. Relation R_3 **is not** reflexive because $(2,2) \notin R_3$.

Examples

Example

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1. $R_1 = \{(1,1), (1,3), (2,1), (2,2), (3,3), (4,2), (4,3), (4,4)\}$.
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3. Relation R_3 **is not** reflexive because $(2,2) \notin R_3$. Relation R_3 **is not** irreflexive because it is not true that $(1,1) \notin R_3$.

Exercise: Reflexive and Irreflexive Relations

Exercise

Let $\mathbb{Z}$ be the set of integers, determine whether relations below is reflexive, irreflexive, or not both.

1. aR_1b iff $a \leq b$.
2. aR_2b iff $a = b + 1$.
3. aR_3b iff a divided by b is integer.
4. aR_4b iff $a \geq b^2$.

Solution:

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1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$,

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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
2. R_2 is not reflexive because $0 \neq 0 + 1$,

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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
2. R_2 is not reflexive because $0 \neq 0 + 1$, R_2 is irreflexive because for all $a \in \mathbb{Z}$ we have $a \neq a + 1$ ($a \not{R}_2 a$).

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is reflexive, irreflexive, or not both.

1. aR_1b iff $a \leq b$.
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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
2. R_2 is not reflexive because $0 \neq 0 + 1$, R_2 is irreflexive because for all $a \in \mathbb{Z}$ we have $a \neq a + 1$ ($a \not R_2 a$).
3. R_3 is not reflexive because 0 divided by 0 is not an integer,

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is reflexive, irreflexive, or not both.

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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
2. R_2 is not reflexive because $0 \neq 0 + 1$, R_2 is irreflexive because for all $a \in \mathbb{Z}$ we have $a \neq a + 1$ ($a \not R_2 a$).
3. R_3 is not reflexive because 0 divided by 0 is not an integer, R_3 is not irreflexive because 1 divides 1.

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is reflexive, irreflexive, or not both.

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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
2. R_2 is not reflexive because $0 \neq 0 + 1$, R_2 is irreflexive because for all $a \in \mathbb{Z}$ we have $a \neq a + 1$ ($a \not R_2 a$).
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4. R_4 is not reflexive because $2 \not\geq 2^2$,

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Solution:

1. R_1 is reflexive because for all $a \in \mathbb{Z}$ we have $a \leq a$, R_1 is not irreflexive because $0 \leq 0$.
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3. R_3 is not reflexive because 0 divided by 0 is not an integer, R_3 is not irreflexive because 1 divides 1.
4. R_4 is not reflexive because $2 \not\geq 2^2$, R_4 is not irreflexive because $0 \geq 0^2$.

Contents

Symmetric, Antisymmetric, & Asymmetric

Definition

Let R be a binary relation on A, relation R is:

1. symmetric, if every $a, b \in A$ satisfies: if aRb then bRa, $(\forall a \forall b\,(aRb \rightarrow bRa))$;
2. antisymmetric, if every $a, b \in A$ satisfies if aRb **and** bRa then $a = b$, $(\forall a \forall b\,(aRb \wedge bRa \rightarrow a = b))$;
3. asymmetric, if every $a, b \in A$ satisfies if aRb then $b \not{R} a$, $(\forall a \forall b\,(aRb \rightarrow b \not{R} a))$.

Question:

Symmetric, Antisymmetric, & Asymmetric

Definition

Let R be a binary relation on A, relation R is:

1. symmetric, if every $a, b \in A$ satisfies: if aRb then bRa, $(\forall a \forall b\, (aRb \rightarrow bRa))$;
2. antisymmetric, if every $a, b \in A$ satisfies if aRb **and** bRa then $a = b$, $(\forall a \forall b\, (aRb \wedge bRa \rightarrow a = b))$;
3. asymmetric, if every $a, b \in A$ satisfies if aRb then $b\not{R}a$, $(\forall a \forall b\, (aRb \rightarrow b\not{R}a))$.

Question: Does antisymmetric mean not symmetric?

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3. asymmetric, if every $a, b \in A$ satisfies if aRb then $b \not R a$, $(\forall a \forall b\, (aRb \rightarrow b \not R a))$.

Question: Does antisymmetric mean not symmetric? Does asymmetric mean not symmetric?

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(1,1), (1,2), (2,1), (2,2), (2,4), (4,2), (4,4)\}$.
2. $R_2 = \{(1,1), (2,3), (2,4), (4,2)\}$.
3. $R_3 = \{(1,1), (2,2), (3,3)\}$.
4. $R_4 = \{(1,2), (3,4)\}$.

We have that:

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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \to 2R_11$ and $2R_14 \to 4R_12$ is T.

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Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \rightarrow 2R_11$ and $2R_14 \rightarrow 4R_12$ is T.
   Relation R_1 **is not** antisymmetric because $(1R_12) \wedge (2R_11) \rightarrow (1 = 2)$ is F.

Examples

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Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \to 2R_11$ and $2R_14 \to 4R_12$ is T.
   Relation R_1 **is not** antisymmetric because $(1R_12) \wedge (2R_11) \to (1 = 2)$ is F.
   Relation R_1 **is not** asymmetric because $(1R_12) \to (2\not{R}_11)$ is F.

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(1,1), (1,2), (2,1), (2,2), (2,4), (4,2), (4,4)\}$.
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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \rightarrow 2R_11$ and $2R_14 \rightarrow 4R_12$ is T.
   Relation R_1 **is not** antisymmetric because $(1R_12) \wedge (2R_11) \rightarrow (1 = 2)$ is F.
   Relation R_1 **is not** asymmetric because $(1R_12) \rightarrow (2\not{R}_11)$ is F.
2. Relation R_2 **is not** symmetric because $2R_23 \rightarrow 3R_22$ is F.

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(1,1), (1,2), (2,1), (2,2), (2,4), (4,2), (4,4)\}$.
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3. $R_3 = \{(1,1), (2,2), (3,3)\}$.
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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \rightarrow 2R_11$ and $2R_14 \rightarrow 4R_12$ is T. Relation R_1 **is not** antisymmetric because $(1R_12) \wedge (2R_11) \rightarrow (1 = 2)$ is F. Relation R_1 **is not** asymmetric because $(1R_12) \rightarrow (2\not R_1 1)$ is F.
2. Relation R_2 **is not** symmetric because $2R_23 \rightarrow 3R_22$ is F. Relation R_2 **is not** antisymmetric because $(2R_24) \wedge (4R_22) \rightarrow (2 = 4)$ is F.

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(1,1), (1,2), (2,1), (2,2), (2,4), (4,2), (4,4)\}$.
2. $R_2 = \{(1,1), (2,3), (2,4), (4,2)\}$.
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We have that:

1. Relation R_1 **is** symmetric because $1R_12 \rightarrow 2R_11$ and $2R_14 \rightarrow 4R_12$ is T. Relation R_1 **is not** antisymmetric because $(1R_12) \wedge (2R_11) \rightarrow (1 = 2)$ is F. Relation R_1 **is not** asymmetric because $(1R_12) \rightarrow (2\not{R}_11)$ is F.
2. Relation R_2 **is not** symmetric because $2R_23 \rightarrow 3R_22$ is F. Relation R_2 **is not** antisymmetric because $(2R_24) \wedge (4R_22) \rightarrow (2 = 4)$ is F. Relation R_2 **is not** asymmetric because $2R_24 \rightarrow 4\not{R}_22$ is F.

3. Relation R_3 **is** symmetric because $1R_31 \rightarrow 1R_31$, $2R_32 \rightarrow 2R_32$, and $3R_33 \rightarrow 3R_33$ are T.

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4. Relation R_4 **is not** symmetric because $1R_42 \rightarrow 2R_41$ is F.

3. Relation R_3 **is** symmetric because $1R_31 \rightarrow 1R_31$, $2R_32 \rightarrow 2R_32$, and $3R_33 \rightarrow 3R_33$ are T. Relation R_3 **is** antisymmetric because $(aR_3b) \wedge (bR_3a) \rightarrow (a = b)$ is T for every $a, b \in \{1, 2, 3\}$, because there are no $a \neq b$ satisfies $(aR_3b) \wedge (bR_3a)$. Relation R_3 **is not** asymmetric because $1R_31 \rightarrow 1\not{R}_31$ is F.
4. Relation R_4 **is not** symmetric because $1R_42 \rightarrow 2R_41$ is F. Relation R_4 **is** antisymmetric because $(aR_4b) \wedge (bR_4a) \rightarrow (a = b)$ always T, because $(aR_4b) \wedge (bR_4a)$ always F.

3. Relation R_3 **is** symmetric because $1R_31 \rightarrow 1R_31$, $2R_32 \rightarrow 2R_32$, and $3R_33 \rightarrow 3R_33$ are T. Relation R_3 **is** antisymmetric because $(aR_3b) \wedge (bR_3a) \rightarrow (a = b)$ is T for every $a, b \in \{1, 2, 3\}$, because there are no $a \neq b$ satisfies $(aR_3b) \wedge (bR_3a)$. Relation R_3 **is not** asymmetric because $1R_31 \rightarrow 1\not R_31$ is F.
4. Relation R_4 **is not** symmetric because $1R_42 \rightarrow 2R_41$ is F. Relation R_4 **is** antisymmetric because $(aR_4b) \wedge (bR_4a) \rightarrow (a = b)$ always T, because $(aR_4b) \wedge (bR_4a)$ always F. Relation R_4 **is** asymmetric because $1R_42 \rightarrow 2\not R_41$ and $3R_44 \rightarrow 4\not R_43$ are T.

Exercise: Symmetry, Antisymmetry, Asymmetry

Exercise

Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
2. aR_2b iff $a \leq b$.
3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

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Solution:

1. R_1 is symmetric because for every integers a and b we have:

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Solution:

1. R_1 is symmetric because for every integers a and b we have:
   $aR_1b \Rightarrow$

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Solution:

1. R_1 is symmetric because for every integers a and b we have:
   $aR_1b \Rightarrow a^2 = b^2 \Rightarrow$

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3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

1. R_1 is symmetric because for every integers a and b we have:
   $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow$

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
2. aR_2b iff $a \leq b$.
3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

1. R_1 is symmetric because for every integers a and b we have:
$aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$.

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
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3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$).

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

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Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$). R_1 is not asymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ ($2R_1 - 2$ but $-2\not{R}_12$ is not true).

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
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Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$). R_1 is not asymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ ($2R_1 - 2$ but $-2\not R_12$ is not true).
2. R_2 is not symmetric because $0 \leq 1$ but $1 \not\leq 0$ ($0R_21$ and $1\not R_20$).

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
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Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$). R_1 is not asymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ ($2R_1 - 2$ but $-2\not{R}_12$ is not true).
2. R_2 is not symmetric because $0 \leq 1$ but $1 \not\leq 0$ ($0R_21$ and $1\not{R}_20$). R_2 is antisymmetric because if $a \leq b$ and $b \leq a$ then $a = b$.

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
2. aR_2b iff $a \leq b$.
3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$). R_1 is not asymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ ($2R_1 - 2$ but $-2\not R_12$ is not true).
2. R_2 is not symmetric because $0 \leq 1$ but $1 \not\leq 0$ ($0R_21$ and $1\not R_20$). R_2 is antisymmetric because if $a \leq b$ and $b \leq a$ then $a = b$. R_2 is not asymmetric because $0 \leq 0$.

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Let $\mathbb{Z}$ be the set of integers, determine whether relations below is symmetric, antisymmetric, asymmetric, or none of them are true.

1. aR_1b iff $a^2 = b^2$.
2. aR_2b iff $a \leq b$.
3. aR_3b iff $a < b$.
4. aR_4b iff a divides b.

Solution:

1. R_1 is symmetric because for every integers a and b we have: $aR_1b \Rightarrow a^2 = b^2 \Rightarrow b^2 = a^2 \Rightarrow bR_1a$. R_1 is not antisymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ but $2 \neq -2$ ($2R_1 - 2$ and $-2R_12$ and $2 \neq -2$). R_1 is not asymmetric because $2^2 = (-2)^2$ and $(-2)^2 = 2^2$ ($2R_1 - 2$ but $-2\not R_12$ is not true).
2. R_2 is not symmetric because $0 \leq 1$ but $1 \not\leq 0$ ($0R_21$ and $1\not R_20$). R_2 is antisymmetric because if $a \leq b$ and $b \leq a$ then $a = b$. R_2 is not asymmetric because $0 \leq 0$. Note that if R_2 asymmetric, then for every $a, b \in \mathbb{Z}$ we must have: if aR_2b then $b\not R_2a$, <u>even if</u> $a = b$.

3. R_3 is not symmetric because $1 < 2$ but $2 \not< 1$ ($1 R_3 2$ and $2 \not R_3 1$).

3. R_3 is not symmetric because $1 < 2$ but $2 \not< 1$ ($1R_3 2$ and $2 \not R_3 1$). R_3 is antisymmetric because there are no integers a and b satisfy both $a < b$ ($aR_3 b$) and $b < a$ ($bR_3 a$).

3. R_3 is not symmetric because $1 < 2$ but $2 \nless 1$ ($1 R_3 2$ and $2 \not R_3 1$). R_3 is antisymmetric because there are no integers a and b satisfy both $a < b$ ($a R_3 b$) and $b < a$ ($b R_3 a$). R_3 is asymmetric because if $a < b$ ($a R_3 b$) then obviously $b \nless a$ ($b \not R_3 a$).

3. R_3 is not symmetric because $1 < 2$ but $2 \nless 1$ ($1R_3 2$ and $2\not R_3 1$). R_3 is antisymmetric because there are no integers a and b satisfy both $a < b$ ($aR_3 b$) and $b < a$ ($bR_3 a$). R_3 is asymmetric because if $a < b$ ($aR_3 b$) then obviously $b \nless a$ ($b\not R_3 a$).
4. R_4 is not symmetric because 1 divides 2 but 2 does not divide 1 ($1R_4 2$ and $2\not R_4 1$).

3. R_3 is not symmetric because $1 < 2$ but $2 \nless 1$ ($1R_32$ and $2\not R_31$). R_3 is antisymmetric because there are no integers a and b satisfy both $a < b$ (aR_3b) and $b < a$ (bR_3a). R_3 is asymmetric because if $a < b$ (aR_3b) then obviously $b \nless a$ ($b\not R_3a$).
4. R_4 is not symmetric because 1 divides 2 but 2 does not divide 1 ($1R_42$ and $2\not R_41$). R_4 is not antisymmetric because 1 divides -1 and -1 divides 1 but $1 \neq -1$ ($1R_4-1$ and $-1R_41$ but $1 \neq -1$).

3. R_3 is not symmetric because $1 < 2$ but $2 \not< 1$ ($1R_32$ and $2\not R_3 1$). R_3 is antisymmetric because there are no integers a and b satisfy both $a < b$ (aR_3b) and $b < a$ (bR_3a). R_3 is asymmetric because if $a < b$ (aR_3b) then obviously $b \not< a$ ($b\not R_3 a$).
4. R_4 is not symmetric because 1 divides 2 but 2 does not divide 1 ($1R_42$ and $2\not R_4 1$). R_4 is not antisymmetric because 1 divides -1 and -1 divides 1 but $1 \neq -1$ ($1R_4 - 1$ and $-1R_41$ but $1 \neq -1$). R_4 is not asymmetric because 1 divides -1 and -1 divides 1 ($1R_4 - 1$ but $-1\not R_4 1$ is not true).

Theorem

Every asymmetric relation is irreflexive.

Contents

Transitive Relations

Definition

Let R be a binary relation on A, relation R is transitive, if for every $a, b, c \in A$ we have: if aRb **and** bRc then aRc, $(\forall a \forall b \forall c\,(aRb \wedge bRc \rightarrow aRc))$.

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.
2. $R_2 = \{(1,1), (2,3), (2,4), (4,2)\}$.
3. $R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$.
4. $R_4 = \{(1,2), (3,4)\}$
5. $R_5 = \{(4,4)\}$.

We have:

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.
2. $R_2 = \{(1,1), (2,3), (2,4), (4,2)\}$.
3. $R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$.
4. $R_4 = \{(1,2), (3,4)\}$
5. $R_5 = \{(4,4)\}$.

We have:

1. R_1 is transitive because:

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$.
2. $R_2 = \{(1, 1), (2, 3), (2, 4), (4, 2)\}$.
3. $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
4. $R_4 = \{(1, 2), (3, 4)\}$
5. $R_5 = \{(4, 4)\}$.

We have:

1. R_1 is transitive because:
   $(3R_12) \wedge (2R_11) \rightarrow (3R_11)$ is T

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$.
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3. $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
4. $R_4 = \{(1, 2), (3, 4)\}$
5. $R_5 = \{(4, 4)\}$.

We have:

1. R_1 is transitive because:
   $(3R_12) \wedge (2R_11) \rightarrow (3R_11)$ is T
   $(4R_12) \wedge (2R_11) \rightarrow (4R_11)$ is T

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.
2. $R_2 = \{(1,1), (2,3), (2,4), (4,2)\}$.
3. $R_3 = \{(1,1), (2,2), (3,3), (4,4)\}$.
4. $R_4 = \{(1,2), (3,4)\}$
5. $R_5 = \{(4,4)\}$.

We have:

1. R_1 is transitive because:
   $(3R_12) \wedge (2R_11) \rightarrow (3R_11)$ is T
   $(4R_12) \wedge (2R_11) \rightarrow (4R_11)$ is T
   $(4R_13) \wedge (3R_11) \rightarrow (4R_11)$ is T

Examples

Example

Let $A = \{1, 2, 3, 4\}$, R_1, R_2, R_3, and R_4 be relations on A defined as:

1. $R_1 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$.
2. $R_2 = \{(1, 1), (2, 3), (2, 4), (4, 2)\}$.
3. $R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
4. $R_4 = \{(1, 2), (3, 4)\}$
5. $R_5 = \{(4, 4)\}$.

We have:

1. R_1 is transitive because:
   $(3R_1 2) \wedge (2R_1 1) \to (3R_1 1)$ is T
   $(4R_1 2) \wedge (2R_1 1) \to (4R_1 1)$ is T
   $(4R_1 3) \wedge (3R_1 1) \to (4R_1 1)$ is T
   $(4R_1 3) \wedge (3R_1 2) \to (4R_1 2)$ is T.
   No a, b, c such that $(aR_1 b) \wedge (bR_1 c) \to (aR_1 c)$ is F.

2. R_2 is not transitive because $(4R_22) \wedge (2R_23) \to (4R_22)$ is F, we also have another counterexample, $(4R_22) \wedge (2R_24) \to (4R_24)$ which is F, also for $(2R_24) \wedge (4R_22) \to (2R_22) \equiv \mathrm{F}$.

2. R_2 is not transitive because $(4R_22) \wedge (2R_23) \rightarrow (4R_22)$ is F, we also have another counterexample, $(4R_22) \wedge (2R_24) \rightarrow (4R_24)$ which is F, also for $(2R_24) \wedge (4R_22) \rightarrow (2R_22) \equiv \mathrm{F}$.
3. R_3 transitive because there is no a, b, and c causing $(aR_3b) \wedge (bR_3c) \rightarrow (aR_3c)$ to be F.

2. R_2 is not transitive because $(4R_22) \wedge (2R_23) \rightarrow (4R_22)$ is F, we also have another counterexample, $(4R_22) \wedge (2R_24) \rightarrow (4R_24)$ which is F, also for $(2R_24) \wedge (4R_22) \rightarrow (2R_22) \equiv \mathrm{F}$.
3. R_3 transitive because there is no a, b, and c causing $(aR_3b) \wedge (bR_3c) \rightarrow (aR_3c)$ to be F.
4. R_4 transitive because there is no a, b, and c causing $(aR_4b) \wedge (bR_4c) \rightarrow (aR_4c)$ to be F.

2. R_2 is not transitive because $(4R_22) \wedge (2R_23) \rightarrow (4R_22)$ is F, we also have another counterexample, $(4R_22) \wedge (2R_24) \rightarrow (4R_24)$ which is F, also for $(2R_24) \wedge (4R_22) \rightarrow (2R_22) \equiv \mathrm{F}$.
3. R_3 transitive because there is no a, b, and c causing $(aR_3b) \wedge (bR_3c) \rightarrow (aR_3c)$ to be F.
4. R_4 transitive because there is no a, b, and c causing $(aR_4b) \wedge (bR_4c) \rightarrow (aR_4c)$ to be F.
5. R_5 transitive because there is no a, b, and c causing $(aR_5b) \wedge (bR_5c) \rightarrow (aR_5c)$ to be F.

Transitive Relation Proof

Definition

Let a and b be two integers with $a \neq 0$, integer a divides b (or b is divisible by a) if there exist an integer k such that $ka = b$.

Example

We have:

- 2 divides 6,

Transitive Relation Proof

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We have:

- 2 divides 6, because we have $3 \cdot 2 = 6$,
- -2 divides 6,

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Example

We have:

- 2 divides 6, because we have $3 \cdot 2 = 6$,
- -2 divides 6, because we have $-3 \cdot -2 = 6$,
- -7 divides 0,

Transitive Relation Proof

Definition

Let a and b be two integers with $a \neq 0$, integer a divides b (or b is divisible by a) if there exist an integer k such that $ka = b$.

Example

We have:

- 2 divides 6, because we have $3 \cdot 2 = 6$,
- -2 divides 6, because we have $-3 \cdot -2 = 6$,
- -7 divides 0, because we have $0 \cdot -7 = 0$,
- 11 divides 11,

Transitive Relation Proof

Definition

Let a and b be two integers with $a \neq 0$, integer a divides b (or b is divisible by a) if there exist an integer k such that $ka = b$.

Example

We have:

- 2 divides 6, because we have $3 \cdot 2 = 6$,
- -2 divides 6, because we have $-3 \cdot -2 = 6$,
- -7 divides 0, because we have $0 \cdot -7 = 0$,
- 11 divides 11, because we have $1 \cdot 11 = 11$,
- 6 does not divide 3,

Transitive Relation Proof

Definition

Let a and b be two integers with $a \neq 0$, integer a divides b (or b is divisible by a) if there exist an integer k such that $ka = b$.

Example

We have:

- 2 divides 6, because we have $3 \cdot 2 = 6$,
- -2 divides 6, because we have $-3 \cdot -2 = 6$,
- -7 divides 0, because we have $0 \cdot -7 = 0$,
- 11 divides 11, because we have $1 \cdot 11 = 11$,
- 6 does not divide 3, because there is no $k \in \mathbb{Z}$ satisfies $k \cdot 6 = 3$.

Prove this theorem.

Theorem

Let $\mathbb{Z}$ be the set of integers and R be a relation on $\mathbb{Z}$ with: aRb iff a divides b, for every $a, b \in \mathbb{Z}$. Relation R is transitive.

Proof That Divisibility Is Transitive

Proof

We will prove that

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \rightarrow aRc$, or:

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \rightarrow aRc$, or: if a divides b and b divides c, then a divides c.
Let a divides b, then

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \to aRc$, or: if a divides b and b divides c, then a divides c.
Let a divides b, then there exists $k \in \mathbb{Z}$ such that $k \cdot a = b$.
Let b divides c, then

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \rightarrow aRc$, or: if a divides b and b divides c, then a divides c.
Let a divides b, then there exists $k \in \mathbb{Z}$ such that $k \cdot a = b$.
Let b divides c, then there exists $\ell \in \mathbb{Z}$ such that $\ell \cdot b = c$.
From the two equations above, we have

Proof That Divisibility Is Transitive

Proof

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Let a divides b, then there exists $k \in \mathbb{Z}$ such that $k \cdot a = b$.
Let b divides c, then there exists $\ell \in \mathbb{Z}$ such that $\ell \cdot b = c$.
From the two equations above, we have

$$\ell \cdot b = c$$

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \rightarrow aRc$, or: if a divides b and b divides c, then a divides c.
Let a divides b, then there exists $k \in \mathbb{Z}$ such that $k \cdot a = b$.
Let b divides c, then there exists $\ell \in \mathbb{Z}$ such that $\ell \cdot b = c$.
From the two equations above, we have

$$\begin{aligned} \ell \cdot b &= c \\ \ell \cdot (k \cdot a) &= c \end{aligned}$$

Proof That Divisibility Is Transitive

Proof

We will prove that $aRb \wedge bRc \rightarrow aRc$, or: if a divides b and b divides c, then a divides c.
Let a divides b, then there exists $k \in \mathbb{Z}$ such that $k \cdot a = b$.
Let b divides c, then there exists $\ell \in \mathbb{Z}$ such that $\ell \cdot b = c$.
From the two equations above, we have

$$\begin{aligned} \ell \cdot b &= c \\ \ell \cdot (k \cdot a) &= c \\ (\ell k) \cdot a &= c, \end{aligned}$$

for some $\ell k \in \mathbb{Z}$. So, a divides c. □

Exercise: Transitive Relations

Exercise

Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

1. xR_1y iff $x^2 \leq y^2$.
2. xR_2y iff $x \geq 2y$.
3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

Exercise: Transitive Relations

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Solution:

1. Assume aR_1b and bR_1c,

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4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$.

Exercise: Transitive Relations

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Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

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3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$,

Exercise: Transitive Relations

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3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$, or $a^2 \leq c^2$.

Exercise: Transitive Relations

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Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

1. xR_1y iff $x^2 \leq y^2$.
2. xR_2y iff $x \geq 2y$.
3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$, or $a^2 \leq c^2$. Then aR_1c.

Exercise: Transitive Relations

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Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

1. xR_1y iff $x^2 \leq y^2$.
2. xR_2y iff $x \geq 2y$.
3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$, or $a^2 \leq c^2$. Then aR_1c. So R_1 is transitive.

Exercise: Transitive Relations

Exercise

Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

1. xR_1y iff $x^2 \leq y^2$.
2. xR_2y iff $x \geq 2y$.
3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$, or $a^2 \leq c^2$. Then aR_1c. So R_1 is transitive.
2. We see that $-4R_2 - 2$ (because $-4 \geq 2\,(-2)$ or $-4 \geq -4$) and $-2R_2 - 1$ (because $-2 \geq 2\,(-1)$ or $-2 \geq -2$).

Exercise: Transitive Relations

Exercise

Let $\mathbb{R}$ be the set of real numbers, determine whether relations below are transitive or not.

1. xR_1y iff $x^2 \leq y^2$.
2. xR_2y iff $x \geq 2y$.
3. xR_3y iff $xy \geq 0$.
4. xR_4y iff $x \geq y^2$.

Solution:

1. Assume aR_1b and bR_1c, then we have $a^2 \leq b^2$ and $b^2 \leq c^2$. So, $a^2 \leq b^2 \leq c^2$, or $a^2 \leq c^2$. Then aR_1c. So R_1 is transitive.
2. We see that $-4R_2 - 2$ (because $-4 \geq 2\,(-2)$ or $-4 \geq -4$) and $-2R_2 - 1$ (because $-2 \geq 2\,(-1)$ or $-2 \geq -2$). But $-4 \not{R}_2 - 1$ because ($-4 \ngeq 2\,(-1)$ or $-4 \ngeq -2$). So R_2 is not transitive.

3. We see that $-2 R_3 0$ (because $-2 \cdot 0 \geq 0$) and $0 R_3 3$ (because $0 \cdot 3 \geq 0$).

3. We see that $-2R_30$ (because $-2 \cdot 0 \geq 0$) and $0R_33$ (because $0 \cdot 3 \geq 0$). But $-2\not R_33$ because $-2 \cdot 3 \not\geq 0$.

3. We see that $-2R_30$ (because $-2 \cdot 0 \geq 0$) and $0R_33$ (because $0 \cdot 3 \geq 0$). But $-2\not R_33$ because $-2 \cdot 3 \not\geq 0$. So R_3 is not transitive.

3. We see that $-2 R_3 0$ (because $-2 \cdot 0 \geq 0$) and $0 R_3 3$ (because $0 \cdot 3 \geq 0$). But $-2 \not R_3 3$ because $-2 \cdot 3 \not\geq 0$. So R_3 is not transitive.
4. We see that $\frac{1}{4} R_4 \frac{1}{2}$ (because $\frac{1}{4} \geq \left(\frac{1}{2}\right)^2$) and $\frac{1}{2} R_4 \frac{1}{\sqrt{3}}$ (because $\frac{1}{2} \geq \left(\frac{1}{\sqrt{3}}\right)^2$).

3. We see that $-2R_30$ (because $-2 \cdot 0 \geq 0$) and $0R_33$ (because $0 \cdot 3 \geq 0$). But $-2\not{R}_33$ because $-2 \cdot 3 \not\geq 0$. So R_3 is not transitive.
4. We see that $\frac{1}{4}R_4\frac{1}{2}$ (because $\frac{1}{4} \geq \left(\frac{1}{2}\right)^2$) and $\frac{1}{2}R_4\frac{1}{\sqrt{3}}$ (because $\frac{1}{2} \geq \left(\frac{1}{\sqrt{3}}\right)^2$). But $\frac{1}{4}\not{R}_4\frac{1}{\sqrt{3}}$ (because $\frac{1}{4} \not\geq \left(\frac{1}{\sqrt{3}}\right)^2$).

3. We see that $-2 R_3 0$ (because $-2 \cdot 0 \geq 0$) and $0 R_3 3$ (because $0 \cdot 3 \geq 0$). But $-2 \not R_3 3$ because $-2 \cdot 3 \ngeq 0$. So R_3 is not transitive.
4. We see that $\frac{1}{4} R_4 \frac{1}{2}$ (because $\frac{1}{4} \geq \left(\frac{1}{2}\right)^2$) and $\frac{1}{2} R_4 \frac{1}{\sqrt{3}}$ (because $\frac{1}{2} \geq \left(\frac{1}{\sqrt{3}}\right)^2$). But $\frac{1}{4} \not R_4 \frac{1}{\sqrt{3}}$ (because $\frac{1}{4} \ngeq \left(\frac{1}{\sqrt{3}}\right)^2$). So R_4 is not transitive.

Challenging Problem

Let p and q be two integers with $q \neq 0$, p is divisible by q if q divides p (there exists $k \in \mathbb{Z}$ such that $kq = p$).

Challenging Problem

Check whether these relations are transitive or not:

1. R is a relation on $\mathbb{Z}$ defined as: aRb iff $a - b$ is divisible by 2, for every $a, b \in \mathbb{Z}$.
2. R is a relation on $\mathbb{Z}$ defined as: aRb iff $a - b$ is not divisible by 2, for every $a, b \in \mathbb{Z}$.
3. R is a relation on $\mathbb{Z}$ defined as: aRb iff ab is divisible by 3, for every $a, b \in \mathbb{Z}$.

Contents

Properties of Relation from Its Representation Matrices I

Let A be a set with n elements and R be a relation on A. If $\mathbf{M}_R$ is a representation matrix for R, then $\mathbf{M}_R$ will have properties that reflect the properties of relation R.

1. If R is reflexive, then

$$\mathbf{M}_R = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{bmatrix}.$$

2. If R is irreflexive, then

$$\mathbf{M}_R = \begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & \ddots & \\ * & & & 0 \end{bmatrix}.$$

Properties of Relation from Its Representation Matrices II

3. If R is symmetric, then $\mathbf{M}_R = (\mathbf{M}_R)^T$, or $\mathbf{M}_R$ is a symmetric matrix.
4. If R is antisymmetric and $\mathbf{M}_R = [m_{ij}]$, then for every $i, j \in \{1, 2, \ldots, n\}$ we have: if $i \neq j$, then $m_{ij} = 0$ or $m_{ji} = 0$.
5. If R is asymmetric and $\mathbf{M}_R = [m_{ij}]$, then for every $i, j \in \{1, 2, \ldots, n\}$ we have: if $m_{ij} = 1$ then $m_{ji} = 0$.
6. If R transitive and $\mathbf{M}_R = [m_{ij}]$, then for every $i, j, k \in \{1, 2, \ldots, n\}$ we have: if $m_{ij} = 1$ **and** $m_{jk} = 1$ then $m_{ik} = 1$.

Exercise: Properties of Relation from Its Matrix

Exercise

Let R be a relation on set A with three elements. Suppose $\mathbf{M}_R$ is a representation matrix of R:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check whether

1. R is reflexive? R is irreflexive?
2. R is symmetric? R is antisymmetric? R is asymmetric?
3. R is transitive?

Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive.

Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive. It is obvious that R is not irreflexive because not all diagonal entry of $\mathbf{M}_R$ is 0.

Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive. It is obvious that R is not irreflexive because not all diagonal entry of $\mathbf{M}_R$ is 0.
2. Because $(\mathbf{M}_R)^T = \mathbf{M}_R$, then R is symmetric.

Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive. It is obvious that R is not irreflexive because not all diagonal entry of $\mathbf{M}_R$ is 0.
2. Because $(\mathbf{M}_R)^T = \mathbf{M}_R$, then R is symmetric. Relation R is not antisymmetric because $m_{12} = 1$ and $m_{21} = 1$.

Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive. It is obvious that R is not irreflexive because not all diagonal entry of $\mathbf{M}_R$ is 0.
2. Because $(\mathbf{M}_R)^T = \mathbf{M}_R$, then R is symmetric. Relation R is not antisymmetric because $m_{12} = 1$ and $m_{21} = 1$. Relation R is not asymmetric because $m_{12} = 1$ but $m_{21} \neq 0$.

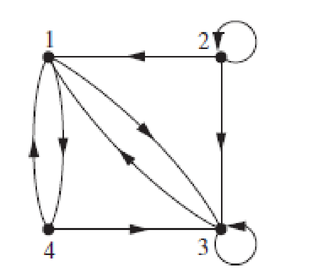
Solution

1. Every diagonal entry of $\mathbf{M}_R$ is 1, then R is reflexive. It is obvious that R is not irreflexive because not all diagonal entry of $\mathbf{M}_R$ is 0.
2. Because $(\mathbf{M}_R)^T = \mathbf{M}_R$, then R is symmetric. Relation R is not antisymmetric because $m_{12} = 1$ and $m_{21} = 1$. Relation R is not asymmetric because $m_{12} = 1$ but $m_{21} \neq 0$.
3. R is not transitive because $m_{12} = 1$ and $m_{23} = 1$ but $m_{13} = 0$.

Exercise

Exercise

Find a representation matrix for relation R whose digraph is



Is R reflexive? irreflexive? symmetric? antisymmetric? asymmetric? transitive?

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R =$

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and not irreflexive (because $m_{22} = 1$).

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and not irreflexive (because $m_{22} = 1$). R is not symmetric (because $m_{21} = 1$ but $m_{12} = 0$),

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and not irreflexive (because $m_{22} = 1$). R is not symmetric (because $m_{21} = 1$ but $m_{12} = 0$), R is not antisymmetric (because $m_{13} = m_{31} = 1$ and $1 \neq 3$),

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and not irreflexive (because $m_{22} = 1$). R is not symmetric (because $m_{21} = 1$ but $m_{12} = 0$), R is not antisymmetric (because $m_{13} = m_{31} = 1$ and $1 \neq 3$), R is not asymmetric (because $m_{13} = 1$ and $m_{31} = 1$).

Solution

Let $\mathbf{M}_R$ be a representation matrix of R, then $\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. R is not reflexive (because $m_{11} = 0$) and not irreflexive (because $m_{22} = 1$). R is not symmetric (because $m_{21} = 1$ but $m_{12} = 0$), R is not antisymmetric (because $m_{13} = m_{31} = 1$ and $1 \neq 3$), R is not asymmetric (because $m_{13} = 1$ and $m_{31} = 1$). R is not transitive (because $m_{14} = 1$ and $m_{41} = 1$ but $m_{11} = 0$).

Contents

Relation Properties from Its Digraph

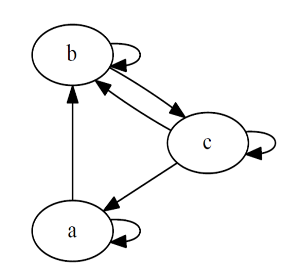
If R is a relation on a finite set A, then some properties of R can be seen from its digraph.

1. R is reflexive iff there exists a loop in every vertex of digraph representation of R.
2. R is irreflexive iff there is no loop in every vertex of digraph representation of R.
3. R is symmetric iff every two vertices in digraph representation of R that have an edge connecting them also have another edge connecting them with opposite direction.
4. R is antisymmetric iff there is no two **different** vertices connected by two edges with opposing direction in digraph representation of R..
5. R is asymmetric iff there is no two vertices connected by two edges with opposing direction **and** there is no loop in every vertex of digraph representation of R.
6. R is transitive iff for every $a, b, c \in V$ we have: if there exists an edge from a to b and from b to c then there exists an edge from a to c.

Exercise

Exercise

Check whether the relation S whose digraph representation denoted below



Digraph for S.

is reflexive; irreflexive; symmetric; antisymmetric; asymmetric; transitive.

Solution to Exercise

Solution: Observe that in digraph representation of S, there are 3 vertices, named a, b, c.

Solution to Exercise

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- S is not symmetric because there is only 1 edge connecting a and b. Relation S is not antisymmetric because b is connected to c with two opposing edges. Relation S is not asymmetric because the digraph representation of S contains loops.
- Relation S is not transitive because there exists an edge from b to c $(b \rightarrow c)$ and from c to a $(c \rightarrow a)$ but there is no edge from b to a $(b \rightarrow a)$,

Contents

Relation Composition (Relation Product)

Definition

Let R be a relation from set A to set B and S be a relation from set B to set C. Composition or product of relations R and S is written as $S \circ R$ and defined as

$$S \circ R = \left\{ \begin{array}{l} (a,c) \mid a \in A, c \in C, \\ \text{there exists } b \in B \text{ that satisfies } (a,b) \in R \text{ and } (b,c) \in S \end{array} \right\}.$$

So $S \circ R$ is a relation from A to C.

Relation Composition Using Arrow Diagram

Relation composition $S \circ R$ can be determined using arrow diagram.

Example

Let $A = \{1, 2, 3\}$, $B = \{2, 4, 6, 8\}$, and $C = \{s, t, u\}$. Let R be a relation from A to B with $R = \{(1, 2), (1, 6), (2, 4), (3, 4), (3, 6), (3, 8)\}$ and S be a relation from B to C with $S = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}$. Arrow diagram representation of $S \circ R$ is

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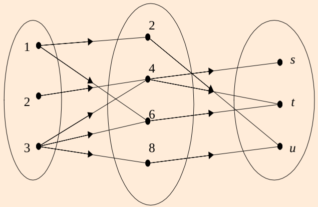

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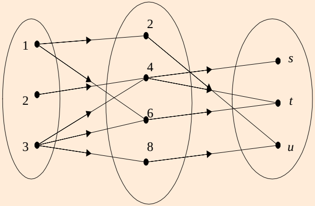

We have $S \circ R = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}$.

Exercise: Relation Composition

Exercise

Let R be a relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ where $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$. Let S be a relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ where $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$. Find $S \circ R$, a relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$.

Solution:

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Let R be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ where $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$. Let S be a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ where $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$. Find $S \circ R$, a relation from $\{1,2,3\}$ to $\{0,1,2\}$.

Solution: Elements of $S \circ R$ are all ordered pair (a,c), satisfying $a \in \{1,2,3\}$, $c \in \{0,1,2\}$, and there exists $b \in \{1,2,3,4\}$ such that aRb and bSc.

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- for $a = 1$, we have $1Rb$ for $b \in \{1, 4\}$.

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- for $a = 2$, we have $2R3$.

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So, $S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$.

Exercise

Let R and S be relations on $A = \{1, 2, 3, 4\}$ defined as: for every $a, b \in A$,

1. aRb iff $b = 5 - a$,
2. aSb iff $a < b$.

Find $S \circ R$.

Solution: Observe that:

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- R relates $a \in A$ and $(5 - a) \in A$.

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- Next, $(5 - a) \in A$ is related by S to elements in A greater than $(5 - a)$.

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- R relates $a \in A$ and $(5 - a) \in A$.
- Next, $(5 - a) \in A$ is related by S to elements in A greater than $(5 - a)$.
- In other words, $S \circ R$ consists of all ordered pair (a, b) with property $b > 5 - a$, or $a + b > 5$, so we have $S \circ R = \{(a, b) \mid a, b \in A$ and

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- Next, $(5 - a) \in A$ is related by S to elements in A greater than $(5 - a)$.
- In other words, $S \circ R$ consists of all ordered pair (a, b) with property $b > 5 - a$, or $a + b > 5$, so we have $S \circ R = \{(a, b) \mid a, b \in A \text{ and } a + b > 5\}$.

Then $S \circ R = \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$.

Associative Property of Relation Composition

Theorem

Relation composition is associative, which means if A, B, C, D are four sets, and

- R is a relation from A to B,
- S is a relation from B to C,
- T is a relation from C to D,

then we have

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

Relation Composition Using Matrix Representation

Definition (*Boolean Product*)

Let $\mathbf{A}$ be a $0-1$ matrix with size $m \times n$ and $\mathbf{B}$ be a $0-1$ matrix with size $n \times p$. *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is defined as an $m \times p$ matrix whose entry in i-th row and j-th column is

$$\mathbf{A} \odot \mathbf{B}[i,j] = \bigvee_{k=1}^{n} \mathbf{A}[i,k] \wedge \mathbf{B}[k,j].$$

.

Boolean product of two matrices is essentially the same with the usual matrix product, with the operation $+$ and $\cdot$ defined as follows:

$+$	$\cdot$
$1+1=1$	$1\cdot 1=1$
$1+0=1$	$1\cdot 0=0$
$0+1=1$	$0\cdot 1=0$
$0+0=0$	$0\cdot 0=0$

Representation Matrix of $S \circ R$

Theorem

Let A, B, C be three finite sets and R be a relation from A to B and S be a relation from B to C. Let $\mathbf{M}_R$ and $\mathbf{M}_S$ be representation matrices for R and S, respectively. If $S \circ R$ is a relation from A to C and $\mathbf{M}_{S \circ R}$ is a representation matrix for $S \circ R$, then

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

Exercise

Exercise

Find a representation matrix of $S \circ R$ if the representation matrices for R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: Let $\mathbf{M}_{S \circ R}$ be a matrix representation for $S \circ R$, then

$$\mathbf{M}_{S \circ R} =$$

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$$\begin{aligned} \mathbf{M}_{S \circ R} &= \mathbf{M}_R \odot \mathbf{M}_S \\ &= \end{aligned}$$

Solution: Let $\mathbf{M}_{S\circ R}$ be a matrix representation for $S \circ R$, then

$$\begin{aligned}
\mathbf{M}_{S\circ R} &= \mathbf{M}_R \odot \mathbf{M}_S \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&=
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&= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0+0+1 & 1+0+0 & 0+0+1 \\ 0+0+0 & 1+0+0 & 0+1+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} \\
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\end{aligned}$$

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&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Exercise

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Let R be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$. Let S be a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ defined as $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$. Find a matrix representation for relation $S \circ R$.

Solution: $\mathbf{M}_R =$

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Solution: $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{M}_S =$

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Let R be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$. Let S be a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ defined as $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$. Find a matrix representation for relation $S \circ R$.

Solution: $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

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Solution: $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S =$$

Exercise

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Let R be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$. Let S be a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ defined as $S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$. Find a matrix representation for relation $S \circ R$.

Solution: $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, $\mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$$\mathbf{M}_{S\circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

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Solution: $\mathbf{M}_R=\begin{bmatrix}1&0&0&1\\0&0&1&0\\1&0&0&1\end{bmatrix}$, $\mathbf{M}_S=\begin{bmatrix}1&0&0\\1&0&0\\0&1&1\\0&1&0\end{bmatrix}$,

$$\mathbf{M}_{S\circ R}=\mathbf{M}_R\odot\mathbf{M}_S=\begin{bmatrix}1&0&0&1\\0&0&1&0\\1&0&0&1\end{bmatrix}\odot\begin{bmatrix}1&0&0\\1&0&0\\0&1&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\1&1&0\end{bmatrix}.$$