Introduction to Tree (Part 1)

Definition of Tree and Some Basic Terminologies about Tree

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret 2 at Fasilkom UI by Team of Lecturers.
6. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to <pleasedontspam>@telkomuniversity.ac.id.

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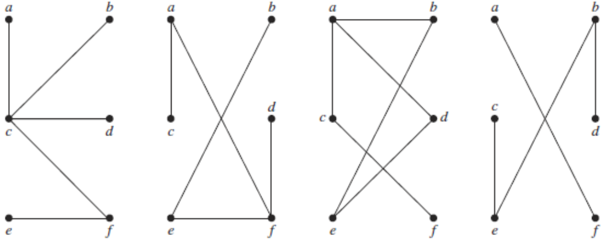
Tree: Definition

Tree

A tree is a **connected** undirected graph that **has no** simple circuit.

Which one of the following graphs are tree?



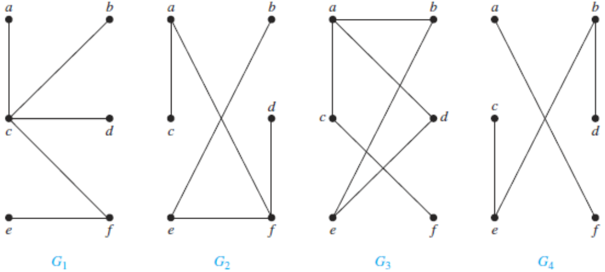
G_1 G_2 G_3 G_4

Tree: Definition

Tree

A tree is a **connected** undirected graph that **has no** simple circuit.

Which one of the following graphs are tree?



Graph G_1 and G_2 are tree, because they are connected and have no simple circuit.

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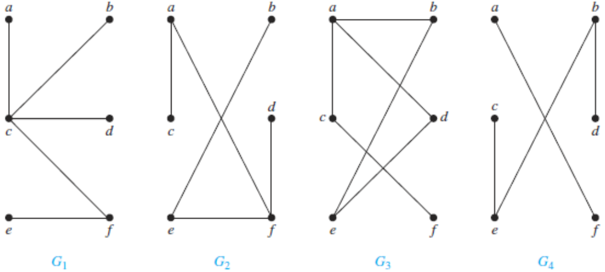

Graph G_1 and G_2 are tree, because they are connected and have no simple circuit.
Graph G_3 is not a tree because it has a simple circuit $\langle a, b, e, d, a \rangle$.

Tree: Definition

Tree

A tree is a **connected** undirected graph that **has no** simple circuit.

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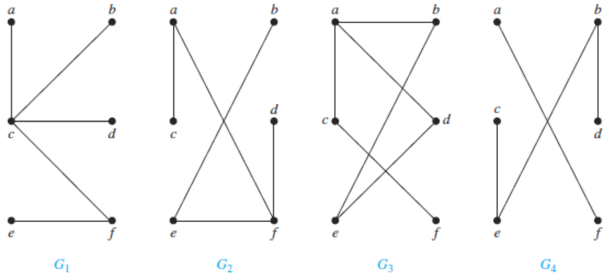


Graph G_1 and G_2 are tree, because they are connected and have no simple circuit. Graph G_3 is not a tree because it has a simple circuit $\langle a, b, e, d, a \rangle$. Graph G_4 is not a tree because it is not connected, there is no path from vertex a to the vertex c.

Theorem

A graph is a tree if and only if every vertex in the graph are connected by a unique simple path.

Definition (Bridge)

An edge on the graph G is called a bridge if the elimination of this edge makes G becomes disconnected.

Theorem

A graph is a tree if and only if every edge in the graph is a bridge.

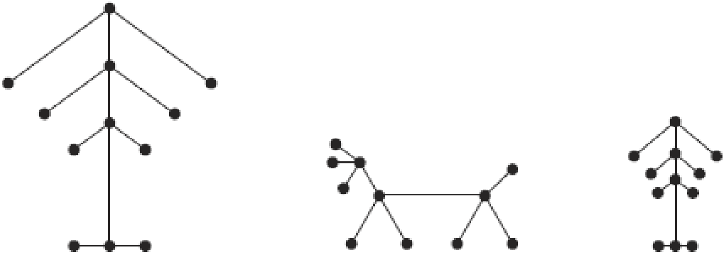
Forest: Definition

Forest

A forest is an undirected graph that **has no** circuit. A forest may contains some trees.

The following example is a forest.

This is one graph with three connected components.



Properties of Tree

Theorem

Suppose $G = (V, E)$ is a simple undirected graph, then the following statements are equivalent:

1. G is a tree,
2. each pair of vertices $u, v \in V$ in G are connected by a unique simple path,
3. G is connected and it satisfies $|E| = |V| - 1$,
4. G has no circuit and it satisfies $|E| = |V| - 1$,
5. G has no circuit and any edge addition in G causes G has exactly one circuit,
6. every edge in G is a bridge.

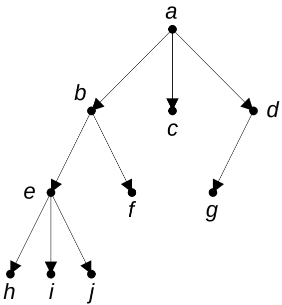
Contents

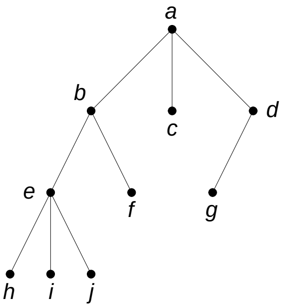
Rooted Tree

Rooted Tree

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root. The direction sign/arrow on a rooted tree can be removed if the root of the tree is clear.

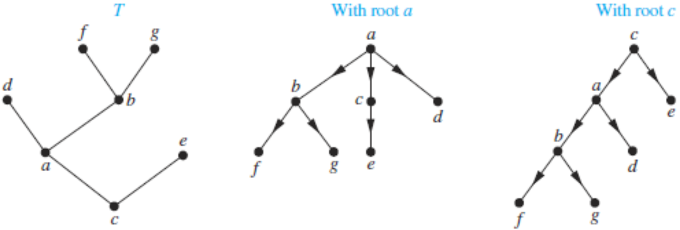
These are rooted trees with root a.



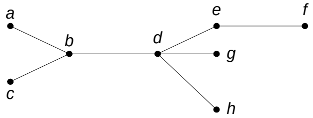


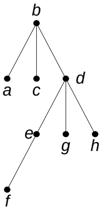
Root Selection

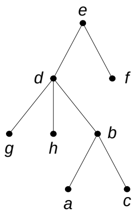
Graphical representations of rooted trees may be different and they depends on the vertex that is chosen as the root.



The middle picture is a rooted tree with root a, whereas rightmost picture is a rooted tree with root c.

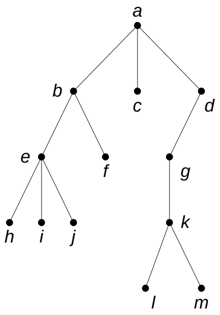






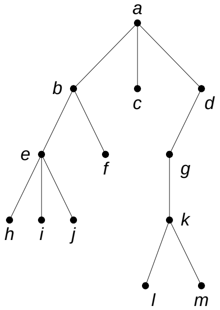
The middle picture is a rooted tree with root b, whereas the rightmost picture is a rooted tree with root e.

Parent, Child/Children, and Path



In the above rooted tree, we have:

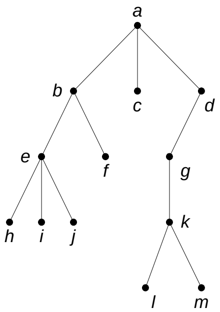
Parent, Child/Children, and Path



In the above rooted tree, we have:

1. the vertices b, c, and d are children of a, each of b, c, and d is a child of a;

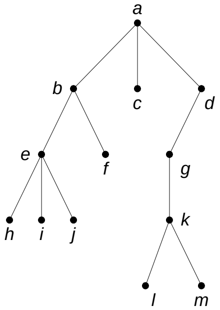
Parent, Child/Children, and Path



In the above rooted tree, we have:

1. the vertices b, c, and d are children of a, each of b, c, and d is a child of a;
2. the vertex a is a parent of b, c, and d;

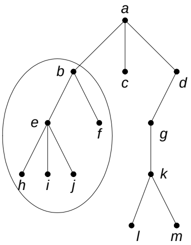
Parent, Child/Children, and Path



In the above rooted tree, we have:

1. the vertices b, c, and d are children of a, each of b, c, and d is a child of a;
2. the vertex a is a parent of b, c, and d;
3. there is only one path from a to j, namely $\langle a, b, e, j \rangle$ of length 3.

Sibling, Cousin, and Subtree



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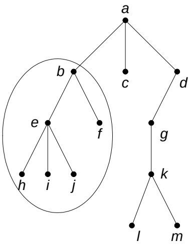
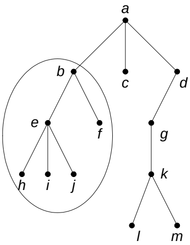


In the above rooted tree, we have:

1. e is a sibling of f and conversely, because e and f have identical parent, namely b;

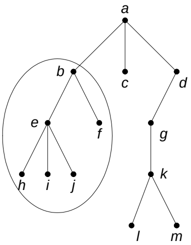
Sibling, Cousin, and Subtree



In the above rooted tree, we have:

1. e is a sibling of f and conversely, because e and f have identical parent, namely b;
2. e is not a sibling of g (or conversely), because they have different parents;

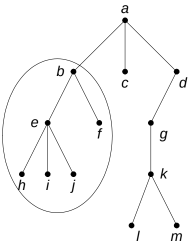
Sibling, Cousin, and Subtree



In the above rooted tree, we have:

1. e is a sibling of f and conversely, because e and f have identical parent, namely b;
2. e is not a sibling of g (or conversely), because they have different parents;
3. although e is not a sibling of g, we say e is a cousin of g, because the parent of e and the parent of g have identical parent, namely a;

Sibling, Cousin, and Subtree



In the above rooted tree, we have:

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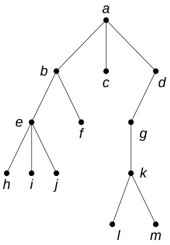
2. e is not a sibling of g (or conversely), because they have different parents;
3. although e is not a sibling of g, we say e is a cousin of g, because the parent of e and the parent of g have identical parent, namely a;
4. the subgraph in the circle with "root" b is called a subtree of a tree with root a.

Vertex Degree in a Rooted Tree

Definition (Degree in Rooted Tree)

Degree of a vertex in a rooted tree is the number of child or the number of subtree of that vertex. Degree of a tree is the largest degree of all degrees of vertices.



In the above rooted tree, we have:

1. $\deg(a) =$

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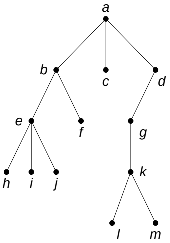

In the above rooted tree, we have:

1. $\deg(a) = 3$, $\deg(b) =$

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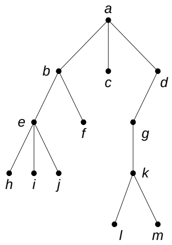

In the above rooted tree, we have:

1. $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) =$

Vertex Degree in a Rooted Tree

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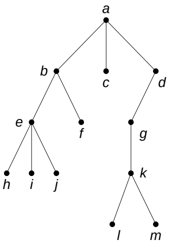

In the above rooted tree, we have:

1. $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 0$, and $\deg(d) =$

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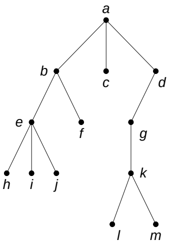

In the above rooted tree, we have:

1. $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 0$, and $\deg(d) = 1$, so here the degree is the out degree (directed downward);

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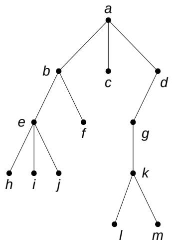


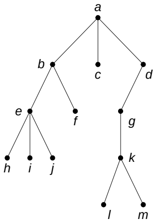
In the above rooted tree, we have:

1. $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 0$, and $\deg(d) = 1$, so here the degree is the out degree (directed downward);
2. the degree of the above tree is 3 because the largest degree of all degrees of vertices is 3 (for vertices a and e).

Leaf and Internal Node

Definition (Leaf and Internal node)

Leaf is a vertex of degree zero (a vertex that has no child). Internal node/ internal vertex is a vertex which is not a root that has child/children (the vertex that has child/children and a parent).



In the above rooted tree:

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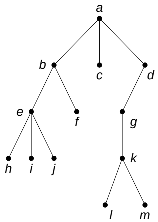

In the above rooted tree:

1. vertices h, i, j, f, c, l, and m are leaves because these vertices have no child,

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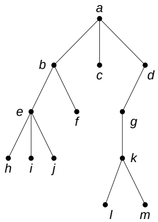


In the above rooted tree:

1. vertices h, i, j, f, c, l, and m are leaves because these vertices have no child,
2. vertices b, d, e, g, and k are internal nodes/ internal vertices because the vertices have child/children and a parent.

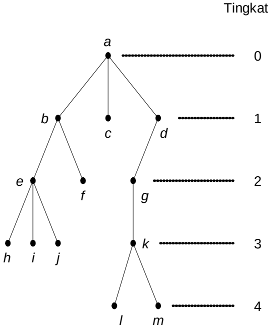
Level and Height/Depth

Definition (Level)

The level of a vertex is the number of edges on a unique path between such a vertex and a root.

Definition (Height or depth)

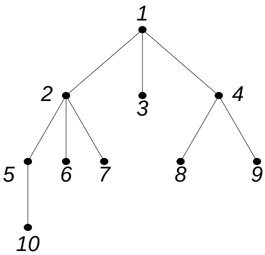
The height or depth of a tree is the largest level that may exist in such a tree.

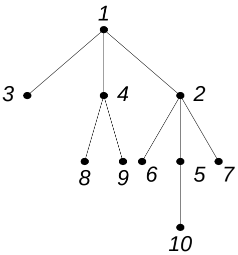


The above rooted tree has four levels.

Ordered Tree

An ordered tree is a tree in which the children of each internal vertex are linearly ordered. An example of ordered tree is as follows.





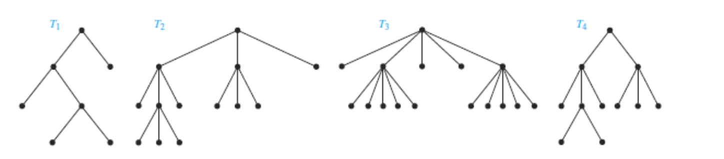
m-ary Tree and Full/Regular m-ary Tree

Definition (m-ary tree and full/regular m-ary tree)

A tree is called an m-ary tree if its root and internal vertices have no more than m children. A full/regular m-ary tree is a tree whose root and internal vertices have exactly m children. A 2-ary tree is called a binary tree.

Exercise

Which one of the following trees are m-ary tree? Determine the value of m!



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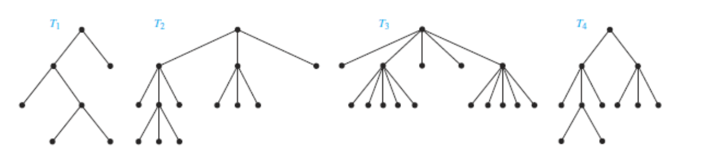
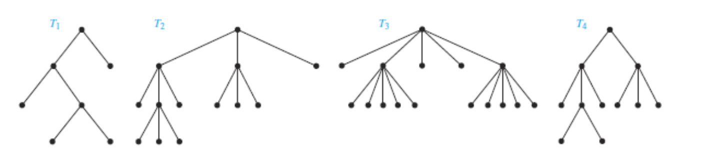


1. T_1 is a full/regular binary tree because the root and every vertex in the tree has exactly 2 children.

Exercise

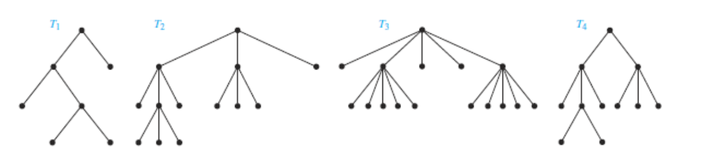
Which one of the following trees are m-ary tree? Determine the value of m!



1. T_1 is a full/regular binary tree because the root and every vertex in the tree has exactly 2 children.
2. T_2 is a full/regular 3-ary tree because the root and every vertex in the tree has exactly 3 children.

Exercise

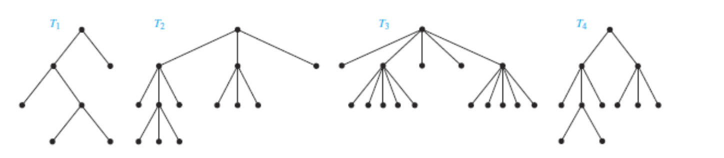
Which one of the following trees are m-ary tree? Determine the value of m!



1. T_1 is a full/regular binary tree because the root and every vertex in the tree has exactly 2 children.
2. T_2 is a full/regular 3-ary tree because the root and every vertex in the tree has exactly 3 children.
3. T_3 is a full/regular 5-ary tree because the root and every vertex in the tree has exactly 5 children.

Exercise

Which one of the following trees are m-ary tree? Determine the value of m!



1. T_1 is a full/regular binary tree because the root and every vertex in the tree has exactly 2 children.
2. T_2 is a full/regular 3-ary tree because the root and every vertex in the tree has exactly 3 children.
3. T_3 is a full/regular 5-ary tree because the root and every vertex in the tree has exactly 5 children.
4. T_4 is a 3-ary tree, but it is not a full/regular 3-ary tree because its root only has 2 children and there is an internal vertex that only has 2 children.

Balanced m-ary Tree

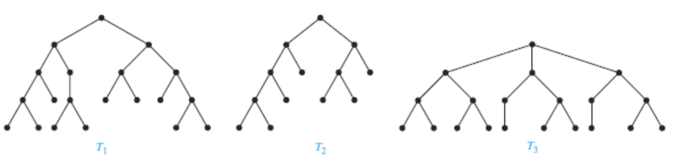
Definition

An m-ary rooted tree with h levels is called a balanced m-ary *tree* if all of its leaves are at level h or $h-1$.

This means that in a balanced m-ary tree all of its leaves are at the same level or at most one level apart.

Exercise

Which one from the following trees are balanced m-ary tree?



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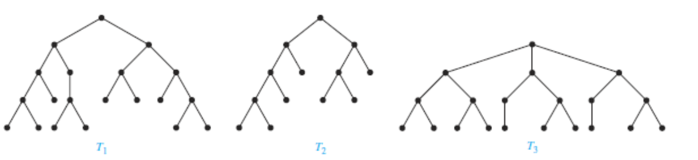
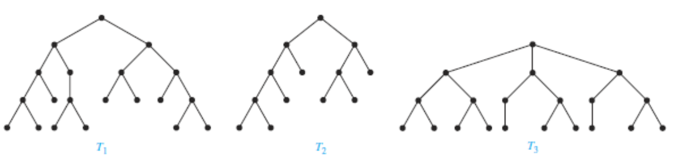


1. T_1 is a balanced binary tree because all of its leaves are at level 4 or 3.

Exercise

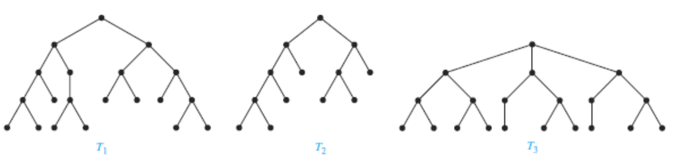
Which one from the following trees are balanced m-ary tree?



1. T_1 is a balanced binary tree because all of its leaves are at level 4 or 3.
2. T_2 is not a balanced binary tree because its leaves are at level 4, 3, or 2.

Exercise

Which one from the following trees are balanced m-ary tree?



1. T_1 is a balanced binary tree because all of its leaves are at level 4 or 3.
2. T_2 is not a balanced binary tree because its leaves are at level 4, 3, or 2.
3. T_3 is a balanced 3-ary tree because all of its leaves are at level 3.

Some Theorems Regarding m-ary Tree

Theorem

There are maximum m^h leaves in an m-ary tree of height h. If the tree is a full/regular tree, the number of its leaves is at most m^h.

Theorem

The maximum number of vertices at the level t of a full/regular m-ary tree is m^t.

Theorem

The maximum number of all vertices of a full/regular m-ary tree of height h is

$$m^0 + m^1 + m^2 + \cdots + m^{h-1} + m^h =$$

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Theorem

The maximum number of all vertices of a full/regular m-ary tree of height h is

$$m^0 + m^1 + m^2 + \cdots + m^{h-1} + m^h = \frac{m^{h+1} - 1}{m - 1}.$$