Introduction to Proof Methods Part 2: Indirect Proof Using Contradiction, Equivalent Statements, and Counterexamples Mathematical Logic – First Term 2022-2023

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School of Computing Telkom University

SoC Tel-U

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Acknowledgements

This slide is compiled using the materials in the following sources:

- Discrete Mathematics and Its Applications (Chapter 1), 8th Edition, 2019, by K. H. Rosen (primary reference).
- Discrete Mathematics with Applications (Chapter 4), 5th Edition, 2018, by S. S. Epp.
- O Discrete Mathematics 1 (2012) slides at Fasilkom UI by B. H. Widjaja.
- Solution Discrete Mathematics 1 (2010) slides at Fasilkom UI by A. A. Krisnadhi.

Some figures are excerpted from those sources. This slide is intended for internal academic purpose in SoC Telkom University. No slides are ever free from error nor incapable of being improved. Please convey your comments and corrections (if any) to <pleasedontspam>@telkomuniversity.ac.id.

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- Proofs of Equivalences
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- 4 Elementary Proofs Strategy
- 5 Mistakes in Proofs

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Theorem

There exists no largest integer.

How do we proof above theorem?

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How do we proof above theorem? The above statement cannot be proven using direct proof nor indirect proof by contraposition.

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- **o** consequently, we infer that $\neg p \rightarrow (r \land \neg r)$ is **true**
- **(**) because $\neg p \rightarrow (r \land \neg r)$ is **true**, and $(r \land \neg r)$ is **false**, we infer that

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- **o** consequently, we infer that $\neg p \rightarrow (r \land \neg r)$ is **true**
- because ¬p → (r ∧ ¬r) is true, and (r ∧ ¬r) is false, we infer that ¬p is false; since ¬p is false, we conclude that p is true.

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There exists no largest integer.

Proof (Sketch)

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- Since M is the largest integer, then $M \ge n$ for all integers n.
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Proof

Suppose there exists a largest integer M. Since M is the largest integer, then $M \ge n$ for all integers n. Let N = M + 1, N is an integer and N > M. This contradicts to our supposition that M is the largest integer. Therefore, there exists no largest integer.

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Exercise 4

Theorem (Theorem 4.1)

Among 37 people in one group, at least four of them born in the same month.

Theorem (Theorem 4.2)

There is no integer which is even and odd simultaneously.

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We have $12 \ \mathrm{months}$ in a year.

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We have 12 months in a year. Suppose the statement that at least four people in the group of 37 people born in the same month is false, then we have

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We have 12 months in a year. Suppose the statement that at least four people in the group of 37 people born in the same month is false, then we have at most three people in the group of 37 people born in the same month.

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We have 12 months in a year. Suppose the statement that at least four people in the group of 37 people born in the same month is false, then we have at most three people in the group of 37 people born in the same month. Consequently, the number of people in the group is at most $3 \cdot 12 = 36$, which contradicts to the fact that there are 37 people in the group. Therefore, among 37 people in one group, at least four of them born in the same month.

Proof (Proof of Theorem 4.2)

We have 12 months in a year. Suppose the statement that at least four people in the group of 37 people born in the same month is false, then we have at most three people in the group of 37 people born in the same month. Consequently, the number of people in the group is at most $3 \cdot 12 = 36$, which contradicts to the fact that there are 37 people in the group. Therefore, among 37 people in one group, at least four of them born in the same month.

Proof (Proof of Theorem 4.2)

Suppose m is an integer which is even and odd simultaneously,

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We have 12 months in a year. Suppose the statement that at least four people in the group of 37 people born in the same month is false, then we have at most three people in the group of 37 people born in the same month. Consequently, the number of people in the group is at most $3 \cdot 12 = 36$, which contradicts to the fact that there are 37 people in the group. Therefore, among 37 people in one group, at least four of them born in the same month.

Proof (Proof of Theorem 4.2)

Suppose m is an integer which is even and odd simultaneously, then there exists integers k and ℓ such that $m = 2k = 2\ell + 1$.

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Proof (Proof of Theorem 4.2)

Suppose m is an integer which is even and odd simultaneously, then there exists integers k and ℓ such that $m = 2k = 2\ell + 1$. Hence, we have $2(k - \ell) = 1$, or $k - \ell = \frac{1}{2}$.

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Proof (Proof of Theorem 4.2)

Suppose m is an integer which is even and odd simultaneously, then there exists integers k and ℓ such that $m = 2k = 2\ell + 1$. Hence, we have $2(k - \ell) = 1$, or $k - \ell = \frac{1}{2}$. This cannot be happened because the set of integers is closed under subtraction. Therefore, there is no integer which is even and odd simultaneously.

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Theorem (Theorem 5)

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Suppose there exists a smallest odd integer M. Then M = 2k + 1 for some integer k and $M \le n$ for all odd integers n. Let N = M - 2 = 2k - 1 = 2(k - 1) + 1, N is odd and N < M. This means N is an odd integer smaller than M, and this contradicts to the assumption that M is the smallest odd integer. Therefore, there is no smallest odd integer.

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The above theorem means that $\sqrt{2}$ cannot be expressed in the form of $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

To prove above theorem, we first consider following definition and lemmas.

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Suppose a and b are integers, not both zero. The greatest common divisor of a and b, denoted by gcd(a, b), is defined as the largest integer that divides both a and b.

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In Bahasa Indonesia, you may familiar with the terminology FPB (*faktor persekutuan terbesar*).

For example, we have: gcd(8,4) =

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Lemma (Lemma 6)

If r is a rational number, then r can be expressed in the form of $\frac{a}{b}$ with $\gcd\left(a,b\right)=1.$

For example, $\frac{8}{18}$ can be expressed as

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For example, $\frac{8}{18}$ can be expressed as $\frac{4}{9}$, observe that gcd(4,9) = 1. The rational number $\frac{a}{b}$ where gcd(a,b) = 1 is called the simplest form of such number .

Lemma (Lemma 7)

Let n be an integer, if n^2 is even, then n is even.

Proof

Left as an exercise.

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Proof (Proof that $\sqrt{2}$ is irrational)

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Proof (Proof that $\sqrt{2}$ is irrational)

Suppose $\sqrt{2}$ is not irrational, then $\sqrt{2}$ is rational.

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Proof (Proof that $\sqrt{2}$ is irrational)

Suppose $\sqrt{2}$ is not irrational, then $\sqrt{2}$ is rational. Therefore (according to Lemma 6) there exists integers a and b with $b \neq 0$ and gcd(a, b) = 1 such that $\frac{a}{b} = \sqrt{2}$.

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Suppose $\sqrt{2}$ is not irrational, then $\sqrt{2}$ is rational. Therefore (according to Lemma 6) there exists integers a and b with $b \neq 0$ and gcd(a, b) = 1 such that $\frac{a}{b} = \sqrt{2}$. By squaring both sides of this equation, we have $\frac{a^2}{b^2} = 2$, or equivalently $a^2 = 2b^2$.

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Observe that b^2 is even. Based on Lemma 7, b is also even, so b = 2d, for some integer d.

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Observe that b^2 is even. Based on Lemma 7, b is also even, so b = 2d, for some integer d. As a result, we have $gcd(a, b) = gcd(2c, 2d) \ge 2$,

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Observe that b^2 is even. Based on Lemma 7, b is also even, so b = 2d, for some integer d. As a result, we have $gcd(a, b) = gcd(2c, 2d) \ge 2$, which contradicts to our previous assumption that gcd(a, b) = 1. Therefore, $\sqrt{2}$ is irrational.

Challenging Problems

Exercise

Determine the truth of these following statements. The notation $a \log_b$ denotes a real number (if exists) such that $a^{a \log b} = b$ (e.g.: $2 \log 8 = 3$, $3 \log 9 = 2$).

- \bigcirc $^{2}\log 3$ is irrational.
- \checkmark $\sqrt[3]{2}$ is irrational.

() If a is even and b is odd, then $a \log b$ is irrational.

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- Suppose there is a theorem (or lemma/proposition) in a biconditional statement: "p if and only if q", or $p \leftrightarrow q$.
- The statement $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \land (q \rightarrow p)$, therefore to show that $p \leftrightarrow q$ is true, we can do this by proving that $p \rightarrow q$ is true and $q \rightarrow p$ is true.

Theorem

Suppose n is an integer, then n is odd if and only if 5n + 6 is odd.

Proof

- Suppose there is a theorem (or lemma/proposition) in a biconditional statement: "p if and only if q", or $p \leftrightarrow q$.
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Suppose n is an integer, then n is odd if and only if 5n + 6 is odd.

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 (\Rightarrow) We first show that if n is odd, then 5n + 6 is odd. Assume n is odd, then n = 2k + 1, for some integer k. We have 5n + 6 = 5(2k + 1) + 6 = 2(5k + 5) + 1. Therefore 5n + 6 is odd.

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Proving Equivalent Statements

- Suppose there is a theorem (or lemma/proposition) in a biconditional statement: "p if and only if q", or $p \leftrightarrow q$.
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Exercise 6

Exercise

Prove or disprove following statement. Let n be an integer:

- n is odd if and only if 7n + 4 is odd,
- 2 n+5 is even if and only if 3n+2 is odd.

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Recall that in order to show that a statement of the form $\forall x P(x)$ is false, it is sufficient to provide a counterexample, i.e., an element c in the universe of discourse that makes P(c) is false.

Exercise

Prove or disprove following statements:

- If x is a nonzero real number, then $x^2 \ge 1$.
- Suppose x, y, and z are integers. If xy = 0 and yz = 0, then xz = 0.

Solution:

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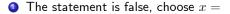
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Prove or disprove following statements:

- If x is a nonzero real number, then $x^2 \ge 1$.
- Suppose x, y, and z are integers. If xy = 0 and yz = 0, then xz = 0.

Solution:

- The statement is false, choose $x = \frac{1}{2}$, then $x^2 = \frac{1}{4} \not\geq 1$.
- The statement is false, choose x = 1, y = 0, and z = 2, then xy = 0, yz = 0, yet xz = 2.

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"Begin at the beginning... and go on till you come to the end: then stop." -Lewis Carrol, Alice's Adventures in Wonderland, 1865

When we need to verify the truth of a particular statement, we can do following steps:

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If the proof methods failed; probably the statement is false and try to guess a counterexample instead.

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What's wrong with these "theorem" and "proof"?



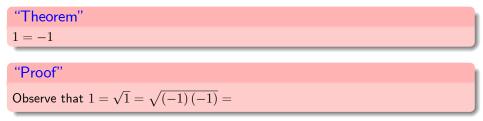
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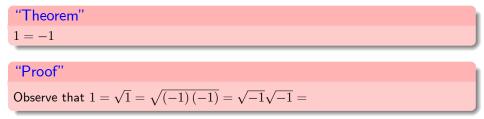


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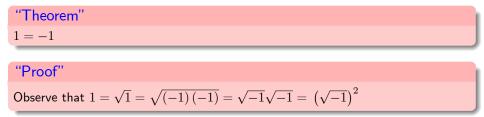
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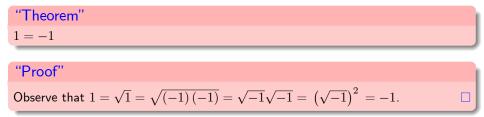
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"Theorem"
$$1 = -1$$

"Proof"

Observe that
$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{-1})^2 = -1.$$

An error occurs in the use of the fact that $\sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$, the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$ can only be used when at least one of a or b is positive. This type of error is an example of *mathematical fallacy*.

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Theorem

If n^2 is even, then n is even.

Proof (?)

Assume that n^2 is even, then $n^2 = 2k$ for some integer k. Suppose $n = 2\ell$ for some integer ℓ , then we have $n^2 = 4\ell^2 = 2(2\ell^2)$. Thus, we conclude that n is even.

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This type of error, which occurs when we use the statement equivalent to the statement being proved (within the prove of itself), is an example of *circular reasoning*.

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