Basic Theory of Graph (Part 3)

Planar Graph and Graph Coloring

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Acknowledgements

This slide is composed based on the following materials:

1. *Discrete Mathematics and Its Applications*, 8th Edition, 2019, by K. H. Rosen (main).
2. *Discrete Mathematics with Applications*, 5th Edition, 2018, by S. S. Epp.
3. *Mathematics for Computer Science*. MIT, 2010, by E. Lehman, F. T. Leighton, A. R. Meyer.
4. Slide for Matematika Diskret 2 (2012). Fasilkom UI, by B. H. Widjaja.
5. Slide for Matematika Diskret 2 at Fasilkom UI by Team of Lecturers.
6. Slide for Matematika Diskret. Telkom University, by B. Purnama.

Some of the pictures are taken from the above resources. This slide is intended for academic purpose at FIF Telkom University. If you have any suggestions/comments/questions related with the material on this slide, send an email to `<pleasedontspam>@telkomuniversity.ac.id`.

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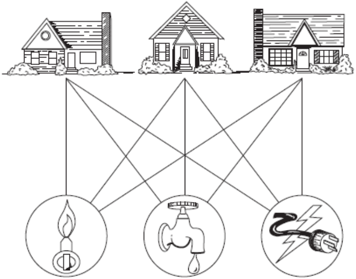
Planar Graph: Motivation

Problems

Suppose there are three houses that must be connected into three utilities, namely: gas, water, and electricity. The three utilities will be connected using their own lines. To avoid fire or any bad things, we must avoid any connection cross (i.e., there cannot be any intersection between two lines for different utilities). Can this idea be implemented?



The problem can be modeled into a bipartite graph $K_{3,3}$.

Planar Graph: Definition

Problems

Given a simple undirected graph $G = (V, E)$, check whether G can be drawn on a plane with no intersection between edges except on its vertices?

Definition (Planar graph)

Planar Graph: Definition

Problems

Given a simple undirected graph $G = (V, E)$, check whether G can be drawn on a plane with no intersection between edges except on its vertices?

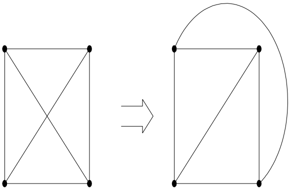
Definition (Planar graph)

Suppose $G = (V, E)$ is a simple undirected graph, G is called a planar graph if G can be drawn on a plane with no edges crossing except on its vertices. The re-drawn of graph G (without edge crossing) is called a planar representation of G. A graph that is not a planar graph is called a non-planar graph.

Contents

Some Examples of Planar Graph

A complete graph K_4 is a planar graph.



Some Examples of Planar Graph

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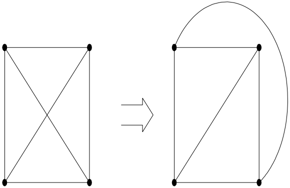
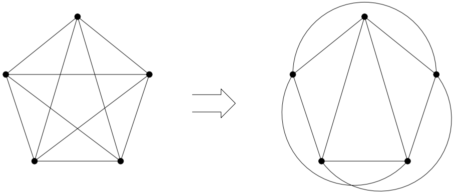


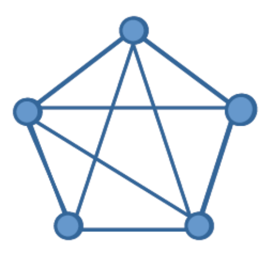
A complete graph K_5 **is not** a planar graph.



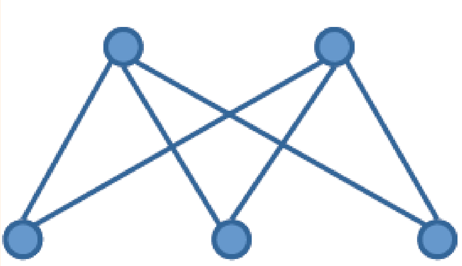
Exercise 1: Planar Graph

Exercise

Check whether the two following graphs are planar graphs or not. Draw their planar representations if possible.



Graph G



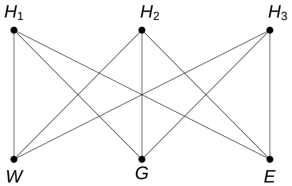
Graph H

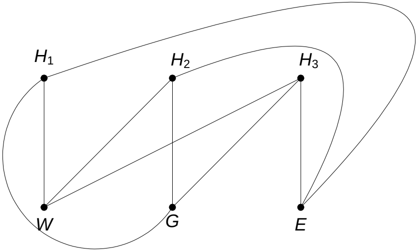
Non-planar Graphs

Are all the graphs planar graphs?

Non-planar Graphs

Are all the graphs planar graphs? Graph $K_{3,3}$ that has been explained previously (the problem of three houses and three utilities) is a non-planar graph.



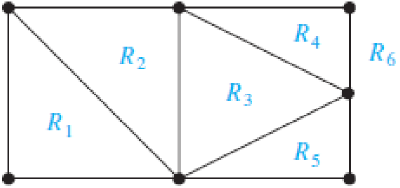


The detailed proof and argument about the non-planarity of $K_{3,3}$ can be read on the textbook.

Contents

Region on Planar Representation of a Graph

Edges of planar representation of a graph on a plane divide the plane into some regions. The regions on the graph can be bounded (which means that the area is limited) or unbounded (which means that the area is unlimited). For example, observe the following graph G.



Graph G

The number of regions on the graph is six, all regions except the region R_6 are bounded.

Euler Formula for Planar Representation of a Graph

Some theorems on planar graphs and planar representation of graphs are as follows. Proof of the theorems can be explored in the textbook or other references.

Theorem

Suppose $G = (V, E)$ is a planar graph and $H = (V, E)$ is a planar representation of G. If r denotes the number of regions on H, then $r = |E| - |V| + 2$.

Theorem (First Euler Inequality for Planar Graph)

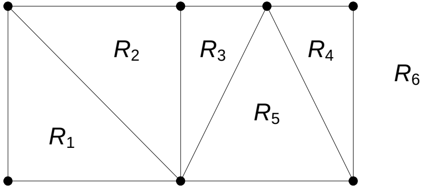
If $G = (V, E)$ is a simple connected graph with **planar** property where $|V| \geq 3$, then $|E| \leq 3|V| - 6$.

Theorem (Second Euler Inequality for Planar Graph)

If $G = (V, E)$ is a simple connected graph with **planar** property where $|V| \geq 3$ and it has no circuit of length 3, then $|E| \leq 2|V| - 4$.

Illustration for the First Equation

Suppose G is the following graph.



We have:

- $r =$

Illustration for the First Equation

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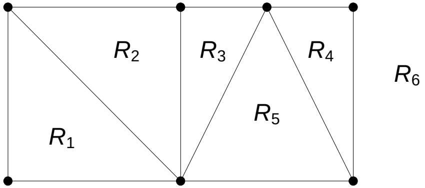

We have:

- $r = 6$ (the number of regions is 6),
- $|E| =$

Illustration for the First Equation

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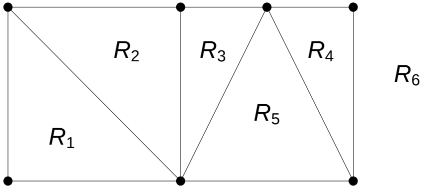

We have:

- $r = 6$ (the number of regions is 6),
- $|E| = 11$ (the number of edges is 11),
- $|V| =$

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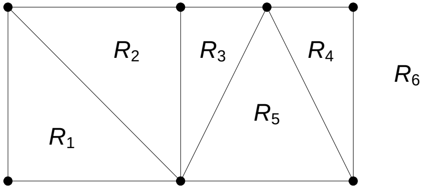

We have:

- $r = 6$ (the number of regions is 6),
- $|E| = 11$ (the number of edges is 11),
- $|V| = 7$.

Notice that

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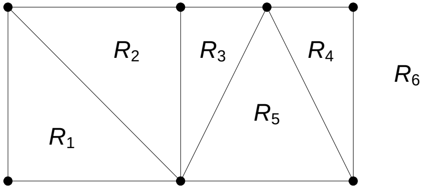

We have:

- $r = 6$ (the number of regions is 6),
- $|E| = 11$ (the number of edges is 11),
- $|V| = 7$.

Notice that

$$\begin{aligned} r &= |E| - |V| + 2 \\ 6 &= 11 - 7 + 2. \end{aligned}$$

Application for the First Euler Inequality

Theorem

K_5 is a non-planar graph.

Proof

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Application for the First Euler Inequality

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Suppose V and E are respectively the set of vertices and the set of edges on K_5. We have $|V| = 5$. Then, because K_5 is a complete graph, then $|E| =$

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Suppose V and E are respectively the set of vertices and the set of edges on K_5. We have $|V| = 5$. Then, because K_5 is a complete graph, then $|E| = \frac{5 \cdot 4}{2} = 10$. Notice that

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The first Euler inequality for planar graph cannot be used to prove that $K_{3,3}$ is a non-planar graph, if $|V|$ and $|E|$ are respectively the set of vertices and the set of edges on $K_{3,3}$, then $|V| =$

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The first Euler inequality for planar graph cannot be used to prove that $K_{3,3}$ is a non-planar graph, if $|V|$ and $|E|$ are respectively the set of vertices and the set of edges on $K_{3,3}$, then $|V| = 6$ and $|E| =$

Application for the First Euler Inequality

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K_5 is a non-planar graph.

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Suppose V and E are respectively the set of vertices and the set of edges on K_5. We have $|V| = 5$. Then, because K_5 is a complete graph, then $|E| = \frac{5 \cdot 4}{2} = 10$. Notice that $|E| \leq 3|V| - 6$ is not satisfied because $10 \not\leq 3 \cdot 5 - 6$. Because K_5 does not satisfy the first Euler inequality for planar graph, then K_5 is not a planar graph. □

The first Euler inequality for planar graph cannot be used to prove that $K_{3,3}$ is a non-planar graph, if $|V|$ and $|E|$ are respectively the set of vertices and the set of edges on $K_{3,3}$, then $|V| = 6$ and $|E| = \frac{6 \cdot 3}{2} = 9$.

Application for the First Euler Inequality

Theorem

K_5 is a non-planar graph.

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Suppose V and E are respectively the set of vertices and the set of edges on K_5. We have $|V| = 5$. Then, because K_5 is a complete graph, then $|E| = \frac{5 \cdot 4}{2} = 10$. Notice that $|E| \leq 3|V| - 6$ is not satisfied because $10 \not\leq 3 \cdot 5 - 6$. Because K_5 does not satisfy the first Euler inequality for planar graph, then K_5 is not a planar graph. □

The first Euler inequality for planar graph cannot be used to prove that $K_{3,3}$ is a non-planar graph, if $|V|$ and $|E|$ are respectively the set of vertices and the set of edges on $K_{3,3}$, then $|V| = 6$ and $|E| = \frac{6 \cdot 3}{2} = 9$. This means $K_{3,3}$ satisfies the first Euler inequality for planar graph, namely $|E| \leq 3|V| - 6$.

Application for the Second Euler Inequality

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Proof

Suppose V and E are respectively the set of vertices and set of edges on $K_{3,3}$. We have $|V| = 6$ and $|E| = 9$. Notice that $K_{3,3}$ has no circuit of length 3 (we can check it through the entries of the main diagonal of the matrix $\mathbf{A}^3_{K_{3,3}}$, all of them are 0).

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Contents

Kuratowski's Theorem: Introduction

In the previous slides, we have already seen that both $K_{3,3}$ and K_5 are non-planar graphs. In this section we will discuss about an important theorem that can be used to check whether a graph has planar properties (efficiently). We first observe the following important facts:

Kuratowski's Theorem: Introduction

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1. Both of $K_{3,3}$ and K_5 are regular graphs, because each of its vertices has identical degree.

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1. Both of $K_{3,3}$ and K_5 are regular graphs, because each of its vertices has identical degree.
2. Both of $K_{3,3}$ and K_5 are non-planar graphs (by Euler's inequality for planar graph).

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2. Both of $K_{3,3}$ and K_5 are non-planar graphs (by Euler's inequality for planar graph).
3. The graph $K_{3,3}$ is a non-planar graph with minimum number of edges, while the graph K_5 is a non-planar graph with minimum number of vertices (the detailed explanation can be read on the textbook or other references).

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1. Both of $K_{3,3}$ and K_5 are regular graphs, because each of its vertices has identical degree.
2. Both of $K_{3,3}$ and K_5 are non-planar graphs (by Euler's inequality for planar graph).
3. The graph $K_{3,3}$ is a non-planar graph with minimum number of edges, while the graph K_5 is a non-planar graph with minimum number of vertices (the detailed explanation can be read on the textbook or other references).
4. Deletion of any edge or any vertex of the graph $K_{3,3}$ or K_5 yields a planar graph (check it by yourself or see the textbook).

Elementary Subdivision

If $G = (V, E)$ is a planar graph and $\{u, v\} \in E$, then the graph $H = (W, F)$ that is obtained by eliminating the edge $\{u, v\}$ and replacing it with the edge $\{u, w\}$ and $\{w, v\}$ (where w is a new vertex) is also a planar graph. Based on these properties we can define the following operation.

Definition (Elementary Subdivision)

Elementary Subdivision

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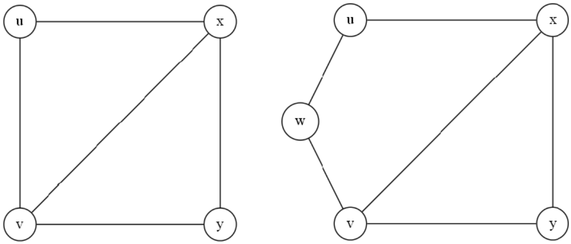
Definition (Elementary Subdivision)

Suppose $G = (V_G, E_G)$ is a simple graph. The graph $H = (V_H, E_H)$ is obtained from an elementary subdivision operation on G if H is obtained by replacing an edge $\{u, v\} \in E_G$ with $\{u, w\}$ and $\{w, v\}$ where w is a new vertex. Formally, the relation between V_H and E_H with V_G and E_H is explained as follows:

1. $V_H = V_G \cup \{w\}$,
2. $E_H = (E_G \setminus \{\{u, v\}\}) \cup \{\{u, w\}, \{w, v\}\}$.

Elementary Subdivision Example

To make it easier, observe the following illustration.



Graf G Graf H

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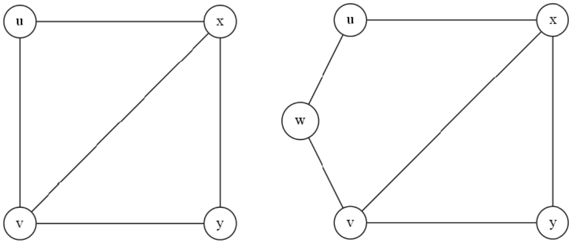

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The graph H is obtained by performing an elementary subdivision operation on graph G. We eliminate the edge

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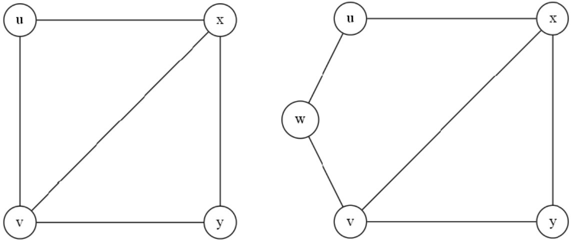

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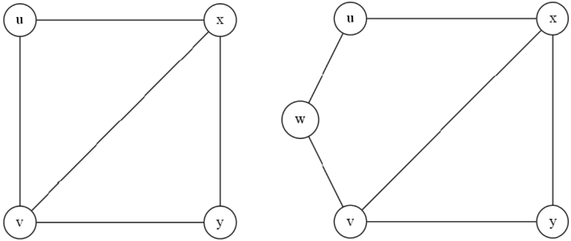

Graf G Graf H

The graph H is obtained by performing an elementary subdivision operation on graph G. We eliminate the edge $\{u, v\}$, add the vertex w, and add the new edges

Elementary Subdivision Example

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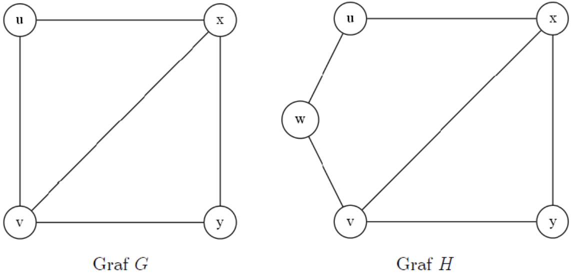
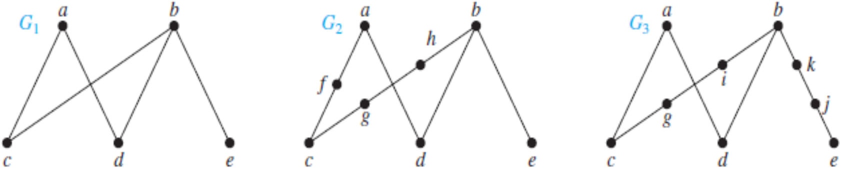


The graph H is obtained by performing an elementary subdivision operation on graph G. We eliminate the edge $\{u, v\}$, add the vertex w, and add the new edges $\{u, w\}$ and $\{w, v\}$.

Two Homeomorphic Graphs

Definition (Two Homeomorphic Graphs)

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two simple graphs, G_1 is called homeomorphic to G_2 if G_1 and G_2 can be obtained from a graph H by applying a sequence of elementary subdivisions (not necessarily identical subdivisions).



Graph G_1, G_2, and G_3

Observe that

1. G_2 can be obtained from G_1 by applying three elementary subdivisions as follows.

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    3. remove the edge $\{g, b\}$, add the vertex h, add the edges $\{g, h\}$ and $\{h, b\}$.

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    3. remove the edge $\{g, b\}$, add the vertex h, add the edges $\{g, h\}$ and $\{h, b\}$.
2. G_3 can be obtained from G_1 by applying four elementary subdivisions as follows.

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    1. remove the edge $\{a, c\}$, add the vertex f, add the edges $\{a, f\}$ and $\{f, c\}$.
    2. remove the edge $\{c, b\}$, add the vertex g, add the edges $\{c, g\}$ and $\{g, b\}$.
    3. remove the edge $\{g, b\}$, add the vertex h, add the edges $\{g, h\}$ and $\{h, b\}$.
2. G_3 can be obtained from G_1 by applying four elementary subdivisions as follows.
    1. remove the edge $\{c, b\}$, add the vertex g, add the edges $\{c, g\}$ and $\{g, b\}$.
    2. remove the edge $\{g, b\}$, add the vertex i, add the edges $\{g, i\}$ and $\{i, b\}$.

Observe that

1. G_2 can be obtained from G_1 by applying three elementary subdivisions as follows.
    1. remove the edge $\{a, c\}$, add the vertex f, add the edges $\{a, f\}$ and $\{f, c\}$.
    2. remove the edge $\{c, b\}$, add the vertex g, add the edges $\{c, g\}$ and $\{g, b\}$.
    3. remove the edge $\{g, b\}$, add the vertex h, add the edges $\{g, h\}$ and $\{h, b\}$.
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    1. remove the edge $\{c, b\}$, add the vertex g, add the edges $\{c, g\}$ and $\{g, b\}$.
    2. remove the edge $\{g, b\}$, add the vertex i, add the edges $\{g, i\}$ and $\{i, b\}$.
    3. remove the edge $\{b, e\}$, add the vertex k, add the edges $\{b, k\}$ and $\{k, e\}$.

Observe that

1. G_2 can be obtained from G_1 by applying three elementary subdivisions as follows.
    1. remove the edge $\{a, c\}$, add the vertex f, add the edges $\{a, f\}$ and $\{f, c\}$.
    2. remove the edge $\{c, b\}$, add the vertex g, add the edges $\{c, g\}$ and $\{g, b\}$.
    3. remove the edge $\{g, b\}$, add the vertex h, add the edges $\{g, h\}$ and $\{h, b\}$.
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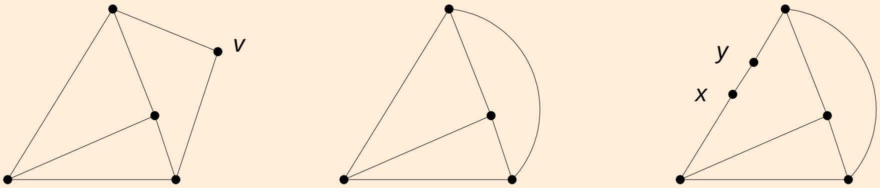
    3. remove the edge $\{b, e\}$, add the vertex k, add the edges $\{b, k\}$ and $\{k, e\}$.
    4. remove the edge $\{k, e\}$, add the vertex j, add the edges $\{k, j\}$ and $\{j, e\}$.

Exercise 2: Homeomorphic Properties

Exercise

Suppose G_1, G_2, and G_3 are the following graphs respectively (from left to right).



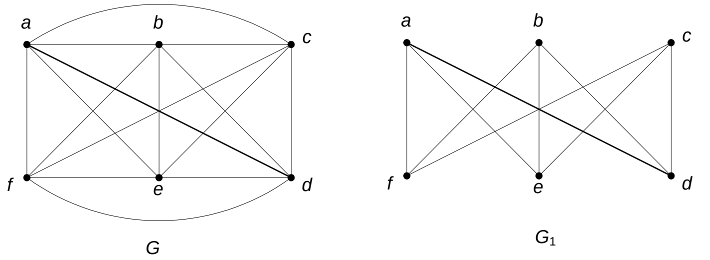
Check whether the three graphs are homeomorphic.

Kuratowski's Theorem

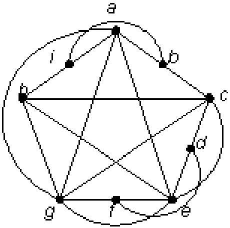
Theorem (Kuratowski's Theorem)

Graph G has non-planar properties if and only if G contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5.

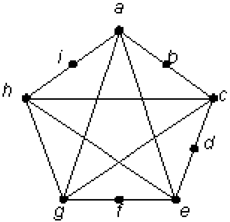
Illustration of Kuratowski's Theorem



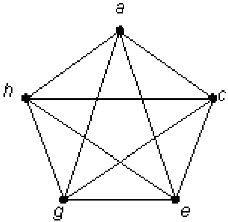
On the above picture, graph G is a non-planar graph because G contains a subgraph G_1 that is isomorphic (and therefore homeomorphic) to $K_{3,3}$.



G



G_1



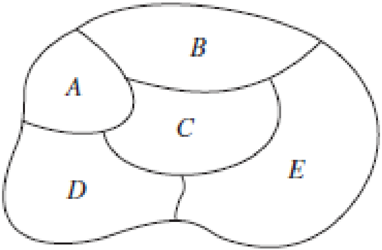
K_5

On the above picture, graph G is a non-planar graph because it contains a subgraph G_1 that is homeomorphic to K_5.

Contents

Graph Coloring: Motivation

Determine the minimum number of colors required to color the following map so that there is no two adjacent regions with the same colour.



Map of a region.

How many different colors required for coloring the map of Indonesia so that there is no two adjacent provinces with the same color?

Graph Coloring: Definition

In graph theory, there are two kinds of graph coloring, namely vertex coloring and edge coloring. We will only discuss the vertex coloring in this course. Hence, the term graph coloring refers to vertex coloring on a graph.

Definition (Graph Colouring)

Graph Coloring: Definition

In graph theory, there are two kinds of graph coloring, namely vertex coloring and edge coloring. We will only discuss the vertex coloring in this course. Hence, the term graph coloring refers to vertex coloring on a graph.

Definition (Graph Colouring)

Suppose $G = (V, E)$ is a simple graph. Graph coloring on G is the assignment of color on the vertices of G such that two **adjacent** vertices on G have **different colors**.

Of course if $G = (V, E)$ and $|V| = n$, then the vertices on G can be colored with n different colours. Indeed, this is not interesting.

Contents

Chromatic Number

Problems

Suppose $G = (V, E)$ is a simple graph. How many minimum colors required for coloring the graph?

Definition (Chromatic Number)

Chromatic Number

Problems

Suppose $G = (V, E)$ is a simple graph. How many minimum colors required for coloring the graph?

Definition (Chromatic Number)

Suppose $G = (V, E)$ is a simple graph, the chromatic number of G, denoted as $\chi(G)$, is defined as the **minimum** number of colors required for coloring the graph.

Four Colors Theorem

Theorem (Four Colours Theorem)

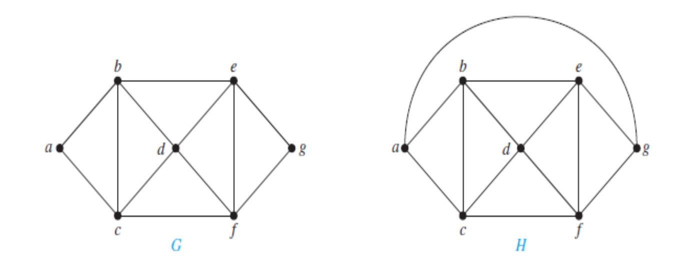
Suppose G is a planar graph, then $\chi(G) \leq 4$.

The proof of the theorem is not easy, you can check the related references on graph coloring.

Exercise 3: Determining the Chromatic Number

Exercise

Determine the chromatic number of the following graphs.



Graph G and H

Solution of Exercise 3

Chromatic number of G, $\chi(G)$, satisfies $\chi(G) \geq 3$, because the vertices a, b, and c must be given different colors. We will show that $\chi(G) = 3$ by the following color assignment:

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1. vertex a is colored red, vertex b is colored blue, and vertex c is colored green,
2. because d is adjacent with vertices b (colored blue) and c (colored green), then d must be red,

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1. vertex a is colored red, vertex b is colored blue, and vertex c is colored green,
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3. because e is adjacent with vertices b (colored blue) and d (colored red), then e must be green,
4. because f is adjacent with vertices c (colored green) and d (colored red), then f must be blue,

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4. because f is adjacent with vertices c (colored green) and d (colored red), then f must be blue,
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4. because f is adjacent with vertices c (colored green) and d (colored red), then f must be blue,
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So we can conclude that $\chi(G) = 3$.

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So we can conclude that $\chi(G) = 3$. The chromatic number for H, $\chi(H)$

Solution of Exercise 3

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3. because e is adjacent with vertices b (colored blue) and d (colored red), then e must be green,
4. because f is adjacent with vertices c (colored green) and d (colored red), then f must be blue,
5. because g is adjacent with vertices e (colored green) and f (green), then g must be red.

So we can conclude that $\chi(G) = 3$. The chromatic number for H, $\chi(H) \geq 4$, because the vertices a, b, c, and g must be given different colors.

Solution of Exercise 3

Chromatic number of G, $\chi(G)$, satisfies $\chi(G) \geq 3$, because the vertices a, b, and c must be given different colors. We will show that $\chi(G) = 3$ by the following color assignment:

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So we can conclude that $\chi(G) = 3$. The chromatic number for H, $\chi(H) \geq 4$, because the vertices a, b, c, and g must be given different colors. The coloring of vertices d, e, f can be done as the coloring for graph G. Therefore, we have $\chi(H) =$

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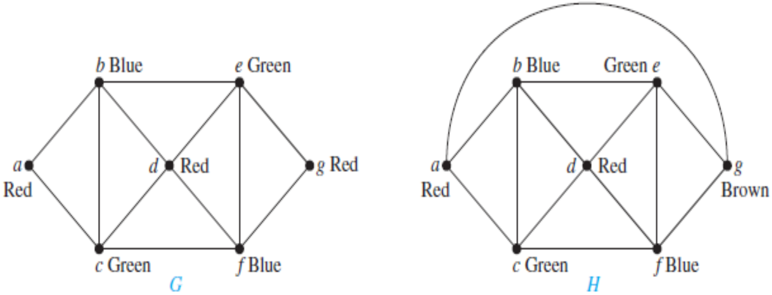
Illustration for Solution of Exercise 3

The illustration of graph coloring is as follows.



Graph coloring for G and H.

Contents

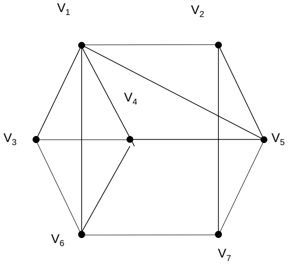
Welsh-Powell Algorithm

Welsh-Powell algorithm is an efficient procedure for coloring a graph.

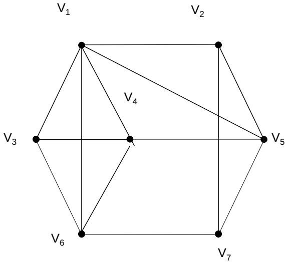
Welsh-Powell Algorithm

1. Suppose $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$.
2. Sort the $\{v_1, v_2, \ldots, v_n\}$ based on their degrees, start from a vertex with the largest degree. The way to sort can be different if there are two vertices or more that have identical degrees.
3. Assign the first color to the vertex with the largest degree. The colorization is performed sequentially so that every vertex in the list that is not adjacent with the previous vertices will be given this color.
4. Repeat step 3 for the vertex with the next largest degree that has not been colored.
5. Repeat step 4 until all vertices have been colored.

Example of Welsh-Powell Algorithm

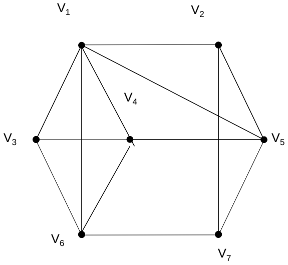


Example of Welsh-Powell Algorithm



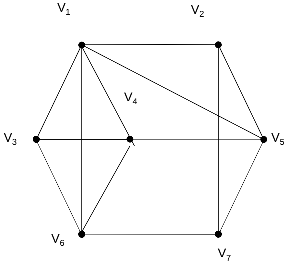
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color							

Example of Welsh-Powell Algorithm



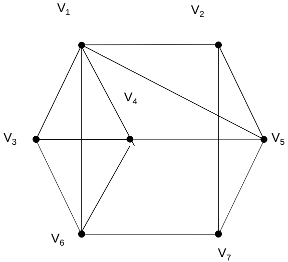
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*						

Example of Welsh-Powell Algorithm



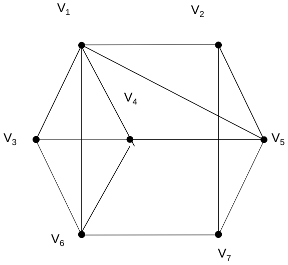
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*					

Example of Welsh-Powell Algorithm



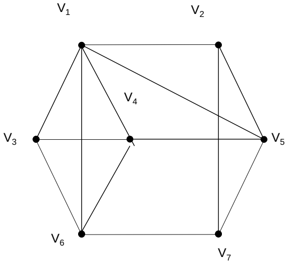
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*	*green*				

Example of Welsh-Powell Algorithm



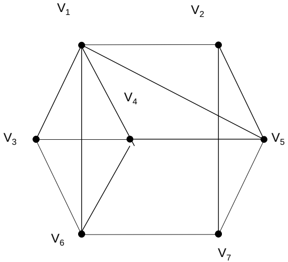
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*	*green*	*green*			

Example of Welsh-Powell Algorithm



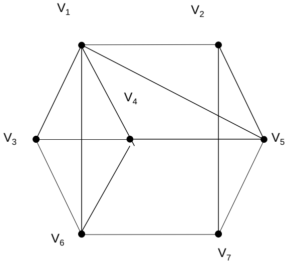
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*	*green*	*green*	*yellow*		

Example of Welsh-Powell Algorithm



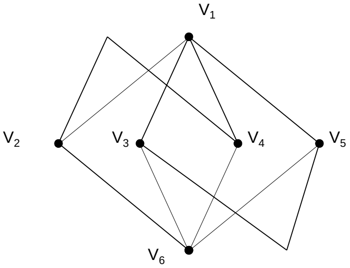
Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*	*green*	*green*	*yellow*	*blue*	

Example of Welsh-Powell Algorithm

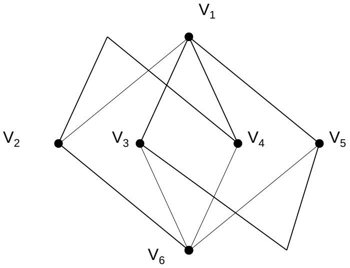


Vertex	v_1	v_4	v_5	v_6	v_2	v_3	v_7
Degree	5	4	4	4	3	3	3
Color	*red*	*yellow*	*green*	*green*	*yellow*	*blue*	*red*

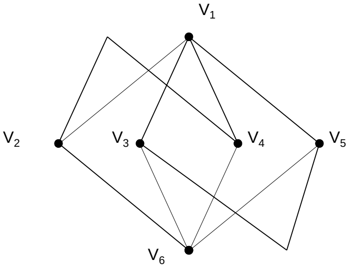
We have $\chi(G) = 4$.



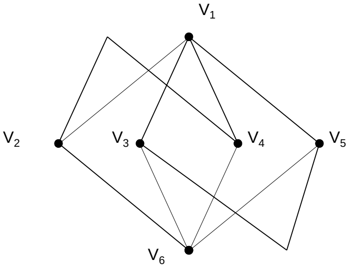
V1
V2
V3
V4
V5
V6



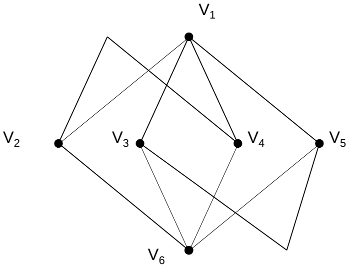
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color						



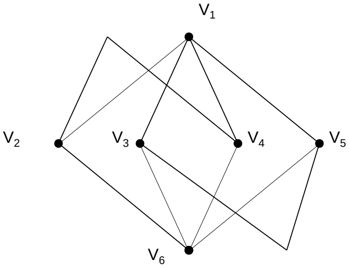
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*					



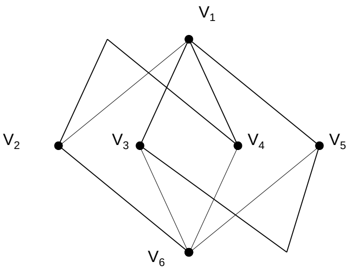
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*	*red*				



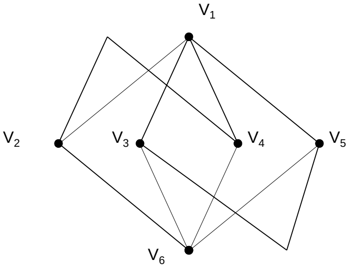
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*	*red*	*yellow*			



Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*	*red*	*yellow*	*yellow*		

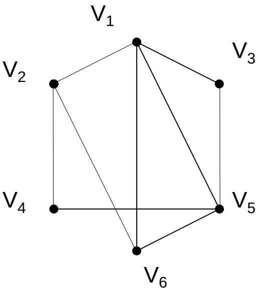


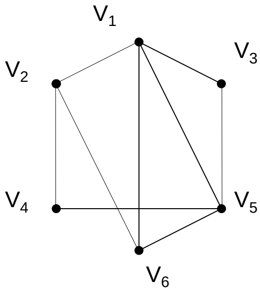
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*	*red*	*yellow*	*yellow*	*green*	



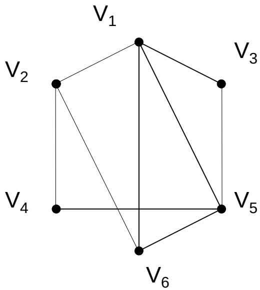
Vertex	v_1	v_6	v_2	v_3	v_4	v_5
Degree	4	4	3	3	3	3
Color	*red*	*red*	*yellow*	*yellow*	*green*	*green*

We have $\chi(G) = 3$.

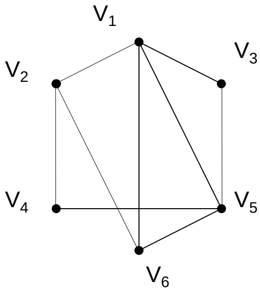

V_1
V_2
V_3
V_4
V_5
V_6



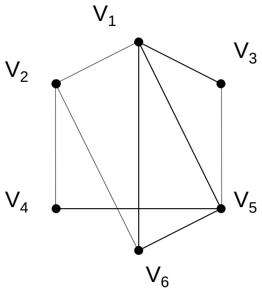
Vertex	v_1	v_5	v_2	v_6	v_3	v_4
Degree	4	4	3	3	2	2
Colour						



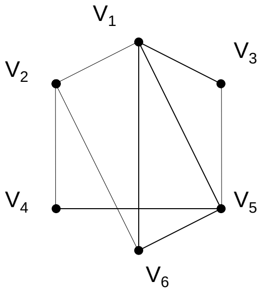
Vertex	v_1	v_5	v_2	v_6	v_3	v_4
Degree	4	4	3	3	2	2
Colour	*red*					



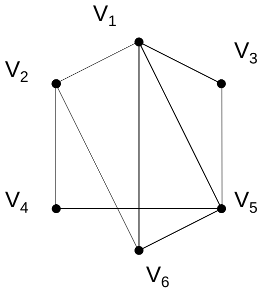
Vertex	v_1	v_5	v_2	v_6	v_3	v_4
Degree	4	4	3	3	2	2
Colour	*red*	*yellow*				



Vertex	v_1	v_5	v_2	v_6	v_3	v_4
Degree	4	4	3	3	2	2
Colour	*red*	*yellow*	*yellow*			



Vertex	v_1	v_5	v_2	v_6	v_3	v_4
Degree	4	4	3	3	2	2
Colour	*red*	*yellow*	*yellow*	*green*		



Vertex	v_1	v_5	v_2	v_6	v_3	v_4
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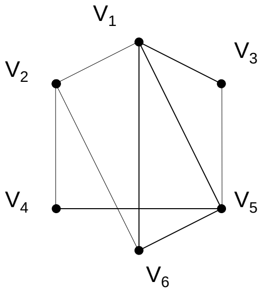
Colour	*red*	*yellow*	*yellow*	*green*	*green*	



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We have $\chi(G) = 3$.

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Determine the minimum number of different slot required so that **there is no** student that has to do two different exams at the same time.

Solution of Problem: Graph Model

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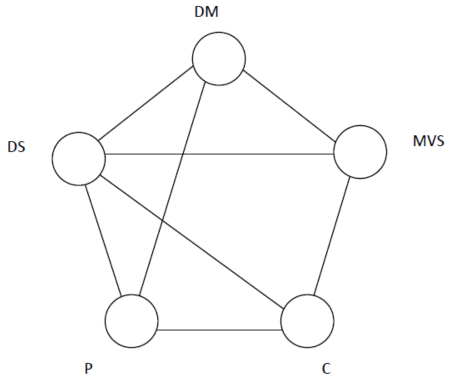

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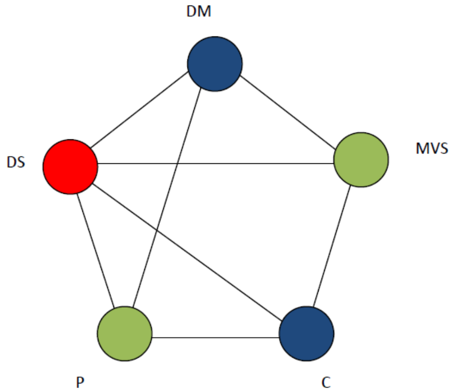

From the previous illustration, we conclude that $\chi(G) =$

From the previous illustration, we conclude that $\chi(G) = 3$. (More detailed argument about this is left as an exercise for the reader.) So there are three different exam slots, with DM exam and C exam are conducted at the same day, and MVS exam and P exam are conducted at the same day as well.